Bonaventura Cavalieri (1598–1647) was one of the most influential mathematicians of his time. He was chiefly noted for his invention of the so-called "Principle of Indivisibles" by which he derived areas and volumes. See pages 134 and 202.   frontis

SOLID GEOMETRY

WITH

PROBLEMS AND APPLICATIONS

REVISED EDITION

BY

H. E. SLAUGHT, Ph.D., Sc.D.

PROFESSOR OF MATHEMATICS IN THE UNIVERSITY OF CHICAGO

AND

N. J. LENNES, Ph.D.

PROFESSOR OF MATHEMATICS IN THE UNIVERSITY
OF MONTANA

ALLYN AND BACON

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PREFACE

In re-writing the Solid Geometry the authors have consistently carried out the distinctive features described in the preface of the Plane Geometry. Mention is here made only of certain matters which are particularly emphasized in the Solid Geometry.

Owing to the greater maturity of the pupils it has been possible to make the logical structure of the Solid Geometry more prominent than in the Plane Geometry. The axioms are stated and applied at the precise points where they are to be used. Theorems are no longer quoted in the proofs but are only referred to by paragraph numbers; while with increasing frequency the student is left to his own devices in supplying the reasons and even in filling in the logical steps of the argument. For convenience of reference the axioms and theorems of plane geometry which are used in the Solid Geometry are collected in the Introduction.

In order to put the essential principles of solid geometry, together with a reasonable number of applications, within limited bounds (156 pages), certain topics have been placed in an Appendix. This was done in order to provide a minimum course in convenient form for class use and not because these topics, Similarity of Solids and Applications of Projection, are regarded as of minor importance. In fact, some of the examples under these topics are among the most interesting and concrete in the text. For example, see pages 170–172, 177, 183–184.

The exercises in the main body of the text are carefully graded as to difficulty and are not too numerous to be easily performed. The concepts of three-dimensional space are made

clear and vivid by many simple illustrations and questions under the suggestive headings "Sight Work." This plan of giving many and varied simple exercises, so effective in the Plane Geometry, is still more valuable in the Solid Geometry where the visualizing of space relations is difficult for many pupils.

The treatment of incommensurables throughout the body of this text, both Plane and Solid, is believed to be sane and sensible. In each case, a frank assumption is made as to the existence of the concept in question (length of a curve, area of a surface, volume of a solid) and of its realization for all practical purposes by the approximation process. Then, for theoretical completeness, rigorous proofs of these theorems are given in Appendix III, where the theory of limits is presented in far simpler terminology than is found in current text-books and in such a way as to leave nothing to be unlearned or compromised in later mathematical work.

Acknowledgment is due to Professor David Eugene Smith for the use of portraits from his collection of portraits of famous mathematicians.

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H. E. SLAUGHT
N. J. LENNES

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CHICAGO and MISSOULA,
     May, 1919.

CONTENTS

PORTRAITS AND BIOGRAPHICAL SKETCHES

SOLID GEOMETRY

SOLID GEOMETRY

INTRODUCTION

1. Two-Dimensional Figures. In plane geometry each figure is restricted so that all of its parts lie in the same plane. Such figures are called *two-dimensional figures*.

A figure, all parts of which lie in one straight line, is a *one-dimensional figure*, while a point is of zero dimensions.

2. Three-Dimensional Figures. A figure, not all parts of which lie in the same plane, is a *three-dimensional figure*.

Thus, a figure consisting of a plane and a line not in the plane is a three-dimensional figure because the whole figure does not lie in one plane.

3. Solid Geometry treats of the properties of three-dimensional figures.

4. Representation of a Plane. While a plane is endless in extent in all its directions, it is represented by a parallelogram, or some other limited plane figure.
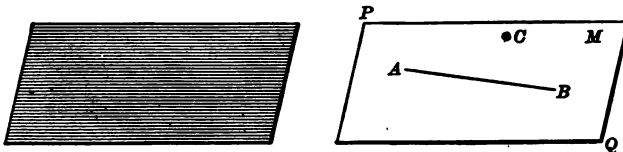
A plane is designated by a single letter in it, by two letters at opposite corners of the parallelogram representing it, or by any three letters in it but not in the same straight line.

Thus, we say *the plane M, the plane PQ,* or *the plane ABC.*

1

5. Figures in Plane and Solid Geometry. In describing a figure in plane geometry, it is assumed, usually without special mention, that all parts of the figure lie in the same plane, while in solid geometry it is assumed that the whole figure need not lie in any one plane.

Thus, in plane geometry we have the theorem :

" *Through a fixed point on a line one and only one perpendicular can be drawn to the line.*"

If all parts of the figure are not required to lie in one plane, the theorem just quoted is far from true. As can be seen from the figure, an unlimited number of lines can be drawn perpendicular to a line at a point in it.
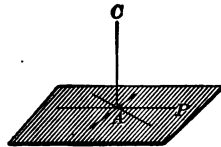
Thus, all the spokes of a wheel may be perpendicular to the axle.

6. Loci in Plane and Solid Geometry. In plane geometry, " the locus of all points at a given distance from a given point " is a circle, while in solid geometry this locus is a sphere.

In plane geometry, " the locus of all points at a given distance from a given line " consists of two lines, each parallel to the given line and at the given distance from it, while in solid geometry this locus is a cylindrical surface whose radius is the given distance.
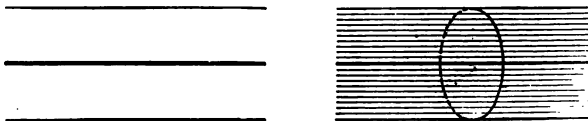
7. Parallel Lines. Skew Lines. In plane geometry, two lines which do not meet are parallel, while in solid geometry, two lines which do not meet need not be parallel. That is, they may not be in the same plane. Lines which are not parallel and do not meet are called *skew* lines.

In solid geometry, as in plane geometry, the definition of parallel lines implies that the lines lie in the same plane. That is, if two lines are parallel, there is always some plane in which both lie. Thus, in the figure, l_1 and l_2 are parallel, as are also l_1 and l_3, while l_3 and l_4 are skew.
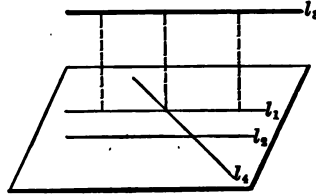
SIGHT WORK

Note. In exercises 1–4 give the required loci for both plane and solid geometry. No proofs are required.

1. The locus of all points six inches distant from a given point.

2. The locus of all points ten inches distant from a given point.

3. The locus of all points at a perpendicular distance of four inches from a given straight line.

4. The locus of all points at a perpendicular distance of nine inches from a given straight line.

5. Find the locus of all points one foot from a given plane. Is this a problem in plane or in solid geometry ?

6. Find the locus of all points equidistant from two parallel lines and in the same plane with them. Is this a problem in plane or in solid geometry ?

7. Find the locus of all points equidistant from two given parallel planes. Is this a problem in plane or in solid geometry ?

8. The side walls of your schoolroom meet each other in four vertical lines. Are any two of these parallel ? Are any three of them parallel ? Do any three of them lie in the same plane ?

9. The side walls of your schoolroom meet the floor and the ceiling in straight lines. Which of these lines are parallel to each other ? Do any of these lines lie in the same plane ?

8. Representation of Solid Figures on a Plane Surface. To represent a figure on a plane surface when at least part of the figure does not lie in that surface requires special devices.

Thus, in the parallelogram *ABCD* used to represent a plane, the edges *AB* and *BC* are made heavier than the other two. This indicates that the lower and right-hand sides are nearer the observer than the other edges. Hence, the plane represented does not lie in the plane of the paper, but the lower part of it stands out toward the observer.
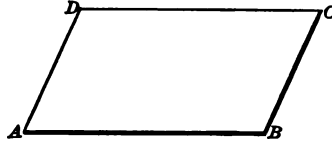
The figure *ABCD* represents a triangular pyramid. The corner marked *B* is nearest the observer and this is indicated by the heavy lines. The triangle *ACD* lies behind the pyramid and is thus farther from the observer. The line *AC* is dotted to indicate that it is seen through the figure.
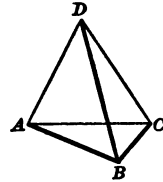
In the closed box *AG*, the lines *AD*, *DC*, and *DH* lie behind the figure and are dotted, while the others are in full view and are solid. If the box were open at the top, part of the line *DH* would be in full view and would be represented by a solid line.
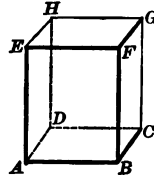
9. Representation of Lines. The following plan for representing lines is generally adhered to in this book:

(1) A line of the main figure which is not obscured by any other part of the figure is represented by a solid line.
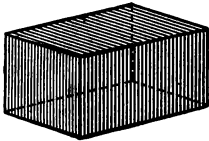
(2) An auxiliary line, which is drawn incidentally in making a proof or constructing a figure, is marked in long dashes if it is in full view.

(3) Any line whatever which is behind a part of the figure is marked in short dashes or dots, or sometimes is not shown at all.

(4) Where a figure is shaded it is usually regarded as opaque and the lines behind it cannot be seen at all.

(5) In some cases a shaded surface is regarded as translucent and the lines behind it are seen dimly. Such lines are marked in short dashes.

The following Axioms and Theorems from plane geometry are referred to in the solid geometry. The special axioms of solid geometry will be given as they arise in the text.

AXIOMS

10. *Things equal to the same things are equal to each other.*

11. *If equals are added to equals, the sums are equal.*

12. *If equals are subtracted from equals, the remainders are equal.*

13. *If equals are multiplied by equals, the products are equal.*

14. *If equals are divided by equals, the quotients are equal.*

15. *If equals are added to unequals, the sums are unequal and in the same order.*

16. *If unequals are added to unequals, in the same order, then the sums are unequal and in that order.*

17. *If equals are subtracted from unequals, the remainders are unequal and in the same order.*

18. *If unequals are subtracted from equals, the remainders are unequal and in the opposite order.*

19. *If a is less than b and b less than c, then a is less than c.*

20. *If a and b are quantities of the same kind, then either $a > b$, or $a = b$, or $a < b$.*

21. *Through a point not on a given line only one straight line can be drawn parallel to that line.*

22. *A straight line-segment is the shortest distance between two points.*

23. *Corresponding parts of equal figures are equal.*

THEOREMS

24. *If two lines intersect, the vertical angles are equal.*

25. *Two triangles are equal if two sides and the included angle of one are equal respectively to two sides and the included angle of the other.*

26. *Two triangles are equal if two angles and the included side of one are equal respectively to two angles and the included side of the other.*

27. *Two triangles are equal if three sides of one are equal respectively to three sides of the other.*

28. *Two points each equidistant from the extremities of a line-segment determine the perpendicular bisector of the segment.*

29. *One and only one perpendicular can be drawn to a line through a point whether that point is on the line or not.*

30. *The sum of all consecutive angles about a point in a plane is four right angles.*

31. *The sum of all consecutive angles about a point and on one side of a straight line is two right angles.*

32. *If two adjacent angles are supplementary, their exterior sides lie in the same straight line.*

33. *If in two triangles two sides of one are equal respectively to two sides of the other, but the third side of the first is greater than the third side of the second, then the included angle of the first is greater than the included angle of the second.*

34. *Two lines which are perpendicular to the same line are parallel.*

35. *If a line is perpendicular to one of two parallel lines, it is perpendicular to the other also.*

36. *If two given lines are perpendicular respectively to each of two intersecting lines, then the given lines are not parallel.*

37. *In a right triangle there are two acute angles.*

38. *From a point in a perpendicular to a straight line, oblique segments are drawn to the line.   Then,*

(1) *If the distances cut off from the foot of the perpendicular are unequal, the oblique segments are unequal, that one being the greater which cuts off the greater distance; and*

(2) *Conversely, if the oblique segments are unequal, the distances cut off are unequal, the greater segment cutting off the greater distance.*

39. *Two angles whose sides are perpendicular, each to each, are equal or supplementary.*

40. *Two right triangles are equal if the hypotenuse and a side of one are equal respectively to the hypotenuse and a side of the other.*

41. *Two right triangles are equal if a side and an acute angle of one are equal respectively to the corresponding side and acute angle of the other.*

42. *Two right triangles are equal if the hypotenuse and an acute angle of one are equal respectively to the hypotenuse and an acute angle of the other.*

43. *A quadrilateral is a parallelogram*
      (1) *if both pairs of opposite sides are equal; or*
      (2) *if two opposite sides are equal and parallel.*

44. *Opposite sides of a parallelogram are equal.*

45. *Two parallelograms are equal if an angle and the two adjacent sides of one are equal respectively to an angle and the two adjacent sides of the other.*

46. *The segment connecting the middle points of the two non-parallel sides of a trapezoid is parallel to the bases and equal to one half their sum.*

47. *The locus of all points equidistant from the extremities of a line-segment is the perpendicular bisector of the segment.*

48. *In the same circle or in equal circles equal chords subtend equal arcs.*

49. *A line perpendicular to a radius at its extremity is tangent to the circle.*

50. *If a line is tangent to a circle, it is perpendicular to the radius drawn to the point of contact.*

51. *If in a proportion the antecedents are equal, then the consequents are equal and conversely.*

52. *In a series of equal ratios the sum of any two or more antecedents is to the sum of the corresponding consequents as any antecedent is to its consequent.*

53. *If a line cuts two sides of a triangle and is parallel to the third side, then any two pairs of corresponding segments form a proportion.*

54. *If two sides of a triangle are cut by a line parallel to the third side, a triangle is formed which is similar to the given triangle.*

55. *In two similar triangles corresponding altitudes are proportional to any two corresponding sides.*

56. *Two triangles are similar if an angle of one is equal to an angle of the other and the pairs of adjacent sides are proportional.*

57. *Two triangles are similar if their pairs of corresponding sides are proportional.*

58. *The area of a parallelogram is equal to the product of its base and altitude.*

59. *Two parallelograms have equal areas if they have equal bases and equal altitudes.*

60. *The area of a triangle is equal to one half the product of its base and altitude.*

61. *If a is a side of a triangle and h the altitude on it and b another side and k the altitude on it, then $ah = bk$.*

62. *The area of a trapezoid is equal to one half the product of its altitude and the sum of its bases.*

63. *The area of a circle is one half the circumference times the radius, or in symbols:*

$$a = \tfrac{1}{2} \cdot 2\,\pi r \cdot r = \pi r^2.$$

BOOK I

PROPERTIES OF THE PLANE

64. Relations of Points, Lines, and Planes. If a line or a plane contains a point, the point is said to be *on* the line or *in* the plane and the line or plane is said to pass *through* the point. If a plane contains a line, the line is said to be *in* the plane and the plane is said to pass *through* the line.

65. AXIOM 1. *If two points of a straight line lie in a plane then the whole line lies in the plane.*

Since a line is endless, it follows from this axiom that a plane is endless in all its directions.

66. AXIOM 2. *Through three non-collinear points one and only one plane can be passed.*

67. AXIOM 3. *Two distinct planes cannot meet in one point only.*

68. Determination of a Plane. A plane is said to be *determined* by certain elements (lines or points) if this plane contains these elements while no other plane does contain them.

While two points determine a straight line it is obvious that two points do not determine a plane. The figure shows three planes, L, M, N, all passing through the two points A and B. But only a certain one of these planes contains a given point C which is not in the line AB.
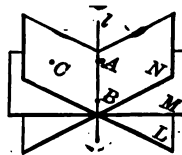
We, therefore, say that three non-collinear points determine a plane, while any number of collinear points fail to determine a plane.

9

LINE COMMON TO TWO PLANES

69. THEOREM I. *Two intersecting planes meet in a straight line.*
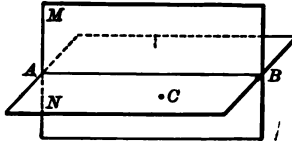
Given two intersecting planes *M* **and** *N*.

To prove that they meet in a straight line AB.

Proof: If two planes intersect they meet in at least two points, as *A* and *B*.   Ax. 3, § 67

But *A* and *B* determine a line which lies wholly in *M* and also wholly in *N*.   Ax. 1, § 65

Hence the planes have the straight line *AB* in common.

A point *C* not in *AB* cannot lie in both *M* and *N*, for in that case the planes would have three non-collinear points in common and hence would coincide.   Ax. 2, § 66

Hence the planes *M* and *N* meet in the straight line *AB*.

                              Q. E. D.

70. Foot of a Line Meeting a Plane. The point in which a straight line meets a plane is called the *foot of the line.*

71. Line and Plane Perpendicular to Each Other. A line is said to be *perpendicular to a plane* if it is perpendicular to every line in the plane passing through its foot. In this case the plane is also said to be perpendicular to the line.

72. Line and Plane Oblique to Each Other. A line which meets a plane and is not perpendicular to it is said to be *oblique* to the plane. The plane is also said to be oblique to the line.

In the figure, *PA* is perpendicular to the plane *M* and *QA* is oblique to it.
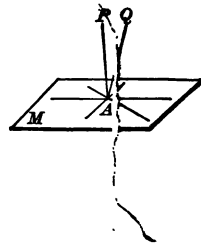
ELEMENTS WHICH DETERMINE A PLANE

73. THEOREM II. *A plane is determined by (1) a line and a point not on it, (2) two intersecting lines, and (3) two parallel lines.*
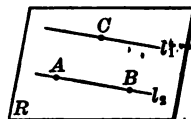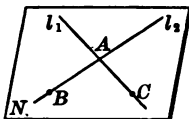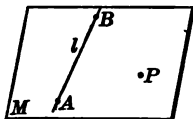
Given (1) a line l and a point P not on it; (2) two intersecting lines l_1 and l_2; (3) two parallel lines l_1 and l_2.

To prove that in each case a plane is determined.

Proof : (1) Let A and B be two points on l. Then one and only one plane M can be passed through l and P because one and only one plane can be passed through A, B, and P. Ax. 2, § 66

(2) Let A be the intersection point of l_1 and l_2, and B and C any other points, one on l_1 and the other on l_2. Then A, B, and C determine the plane N in which lie l_1 and l_2. Axs. 2, 1. §§ 66, 65

(3) By definition l_1 and l_2 lie in a plane R. They lie in only one such plane since the points A and B on l_1 and C on l_2 lie in only one plane. Q. E. D.

74. COROLLARY 1. *Through a line there is more than one plane.*

Suggestion. Let M be a plane through the given line l, and C a point not in M. Then l and C determine a plane N through l different from M.

75. COROLLARY 2. *At a point on a line there is more than one perpendicular to the line.*

Suggestion. Let M and N be planes each passing through the given line l. Then in each plane there is a line $\perp l$ at any point A on it.
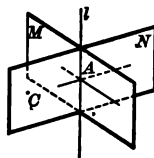
LINE PERPENDICULAR TO THE PLANE OF TWO GIVEN LINES

76. THEOREM III. *If a line is perpendicular to each of two lines at their point of intersection, it is perpendicular to the plane of these lines.*
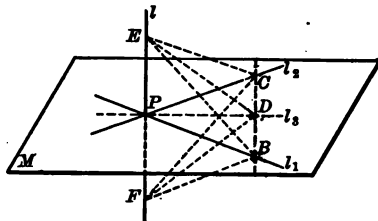
Given a line *l* perpendicular to each of the lines l_1 and l_2 at the point *P*.

To prove that l is perpendicular to the plane of l_1 and l_2.

Proof: Let *M* be the plane of l_1 and l_2, and let l_3 be any line in *M* through *P*. Draw a line meeting l_1, l_2, and l_3 in the points *B*, *C*, and *D* respectively. Let *E* and *F* be points on *l*, on opposite sides of *P*, and such that $EP = FP$. Draw *EB, ED, EC, FB, FD, FC.*

Then prove:

(1) $\triangle EBP = \triangle FBP$; (2) $\triangle ECP = \triangle FCP$;
(3) $\triangle EBC = \triangle FBC$; (4) $\triangle EBD = \triangle FBD$;
(5) $\triangle EPD = \triangle FPD$; (6) $\angle EPD = \angle FPD$.

$\therefore$ *EP* is perpendicular to l_3.   **Why?**

But l_3 is any line in *M* through *P*.

$\therefore$ line $l \perp$ plane *M*.   **§ 71**

Q. E. D.

77. COROLLARY. *If each of two lines is perpendicular to a third line at the same point, then the plane of the two lines is perpendicular to the third line.*

SIGHT WORK

The diagram on this page represents a three-dimensional figure in the shape of an ordinary rectangular box. In this figure the points A, K, and B, for instance, do not determine a plane, since they all lie on the same straight line, while A, B, and C do not lie in a straight line, and hence determine a plane.

1. In this figure pick out several lines which lie in one of the surfaces and are not obscured by the figure.

2. Pick out several lines which are obscured by the figure ; also some which lie within the figure.

3. Pick out four sets of three points each which do not determine planes, and also four sets which do determine planes.
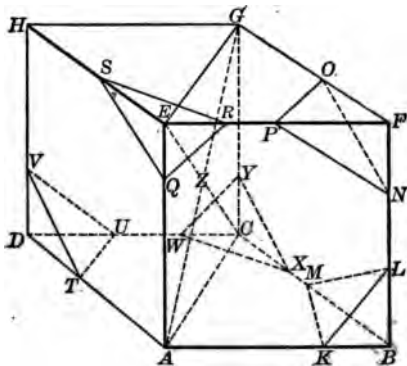
4. Is the line AB perpendicular to the plane BCG? Why? Is AB perpendicular to the plane AEH?

5. Pick out six planes in the figure, each determined by parallel lines.

6. Do the points C, Z, E determine a plane? the points C, Z, G? the points B, F, Z?

7. Using the schoolroom, or a room at home, locate planes corresponding to the planes AEG, KLM, NOP, and EFG, in the above figure.

8. Point out in some room planes determined by points corresponding to D, E, B ; D, F, B ; D, C, F ; A, B, H in the above figure.

PLANE PERPENDICULAR TO A LINE

78. THEOREM IV. *Through a point there is one and only one plane perpendicular to a line.*
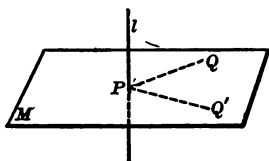
FIG. 1.   FIG. 2.

Given a line *l* and a point *P*.

To prove that through P there is one and only one plane ⊥ l.

Proof: (1) *When the point P is on the line l.*   Fig. 1.

Through *P* draw lines *PQ* and *PQ'* both ⊥ *l.* Then the plane *M*, determined by *PQ* and *PQ'*, is ⊥ *l.*   § 77

To prove that *M* is the only plane through *P* which is ⊥ *l*, suppose that a plane *M'* through *P* is also ⊥ *l.* Let *R* be a plane through *l* meeting *M* and *M'* in two lines. Then these lines would both lie in *R* and be ⊥ *l*, which is impossible. § 29

(2) *When the point P is not on the line l.*   Fig. 2.

Let *PQ* be a line ⊥ *l* and let *N* be a plane ⊥ *l* at *Q*.

Then the plane *N* passes through *P* and is the plane required.

For if *P* does not lie in *N*, then a plane *R'* determined by *l* and *PQ* cuts *N* in a line *l'* which is ⊥ *l* (§ 71), and *PQ* and *l'* are each ⊥ *l* at the point *Q* in plane *R'*, which is impossible.   § 29

Suppose, now, that there are two planes through *P* each ⊥ *l.* These planes cannot meet *l* in the same point (Case 1). Let them meet *l* in *Q* and *Q'*. Then *PQ* and *PQ'* are each ⊥ *l*, which is impossible.   Q. E. D.
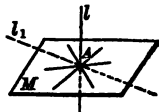
79. COROLLARY. *All lines perpendicular to a line at the same point lie in the plane perpendicular to the line at this point.*

LINE PERPENDICULAR TO A PLANE

**80.** THEOREM V. *Through a point there is one and only one line perpendicular to a plane.*

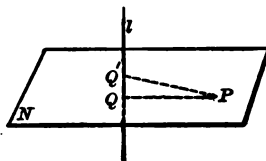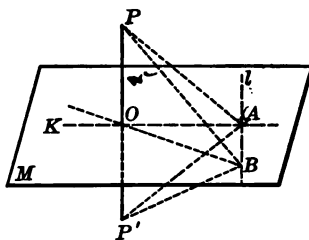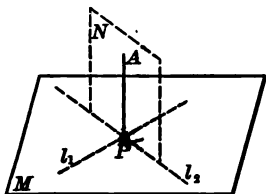**Given a plane *M* and a point *P*.**

To prove that through P there is one and only one line $\perp$ M.

Proof : (1) *When P is in the plane M* (first figure). Let l_1 be any line in *M* through *P*, and let *N* be a plane $\perp l_1$ at *P* and meeting *M* in the line l_2. Let *PA* be a line in *N* and $\perp l_2$. Then *PA* is also $\perp l_1$. § 71

$$\therefore PA \perp M. \qquad \qquad § 76$$

(2) *When P is not in the plane M* (second figure). Let *l* be any line in the plane *M*. Through *P* pass a plane $N' \perp l$ at *A* and meeting *M* in the line *AK*. From *P* in plane *N'* draw a line $PO \perp AK$ and extend it to *P'* so that $OP' = PO$. Let *B* be any point in *l* different from *A*. Draw *PB*, *PA*, *P'B*, *P'A*, and *OB*.

Then prove (1) $\triangle POA = \triangle P'OA$; (2) $\triangle PAB = \triangle P'AB$; (3) $\triangle POB = \triangle P'OB$; (4) $\angle POB = \angle P'OB$; (5) $PO \perp OB$.

$$\therefore PO \perp M. \qquad \qquad § 76$$

If in either case (1) or case (2) there were two lines *PA* and *PB* each $\perp$ *M*, then the plane *R* of these lines would cut *M* in a line *l*. Hence *PA* and *PB* would both lie in *R* and be $\perp l$, which also lies in *R*. But this is impossible by § 29.
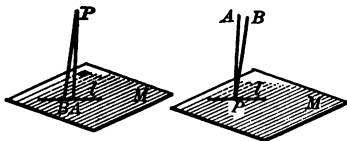
Hence *PA* is the only line through *P* which is $\perp$ *M*. Q. E. D

SIGHT WORK

1. Does a stool with three legs always stand firmly on a flat floor ? Why ?

2. Does a table with four legs always stand firmly on a flat floor ? Why ? On what conditions will such a table stand firmly on a flat floor ?

3. If the point C does not lie in the plane ABD, how many different planes are determined by the points A, B, C, D ?
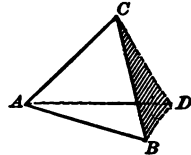
4. How many planes are determined by any four points which do not all lie in one plane ?

5. How many planes are determined by the points A, B, C, D, E, if A, B, C lie in a straight line and C, D, E lie in another straight line ?

6. How many planes are determined by five points, no three of which lie in a straight line and no four of which lie in the same plane ?

7. How many planes are determined by three lines l_1, l_2, l_3 all passing through the same point but not all lying in the same plane ?
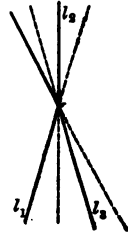
8. How many planes are determined by four lines which all meet in a point, but no three of which lie in the same plane ?

9. How many planes are determined by three lines all parallel to each other, and not all lying in the same plane ?

10. How many planes are determined by four lines all parallel to each other, and no three lying in the same plane ?

11. A line cannot be perpendicular to each of two intersecting planes. Why ?

Suggestions. (1) If a line l is perpendicular to the planes M and N at the points A and B, and C is a point in their intersection, then $\triangle ABC$ would contain two right angles. (2) If l is perpendicular to M and N at a point P in their intersection, pass a plane through l, meeting M and N in l_1 and l_2. Then in this plane l_1 and l_2 are both $\perp l$.
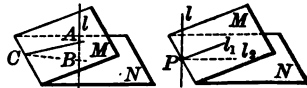
OBLIQUE LINES FROM A POINT TO A PLANE

81. THEOREM VI. *Oblique lines from a point to a plane meeting the plane at equal distances from the foot of the perpendicular are equal; and*

conversely, two equal oblique lines from a point to a plane meet the plane at equal distances from the foot of the perpendicular.
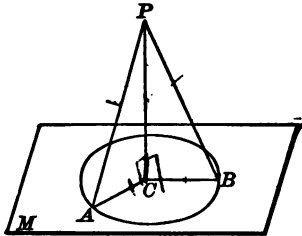
(1) **Given** $PC \perp M$, and $AC = BC$.   *To prove that* $PA = PB$.
Suggestion. Prove $\triangle PCA = \triangle PCB$.

(2) **Given** $PC \perp M$, and $PA = PB$.   *To prove that* $AC = BC$.
Suggestion. Prove $\triangle PCA = \triangle PCB$.

82. COROLLARY. *The perpendicular is the shortest distance from a point to a plane.*

Hence *the distance* from a point to a plane means the perpendicular distance.

SIGHT WORK

Without giving proofs describe the following loci :

1. All points equidistant from the points on a circle.

2. All points equidistant from the vertices of a triangle.

3. All points in a plane which are at a given distance from a given point outside the plane. If a perpendicular be drawn to the plane from this outside point, how is its foot related to this locus ?

EXERCISES

1. Show how a carpenter could use the theorem of § 76 to stand a post perpendicular to the floor, if he has at hand two ordinary steel squares.

2. Show how a back-stop on a ball field can be made per-pendicular to the line through second base and the home plate. What theorems of solid geometry are used?
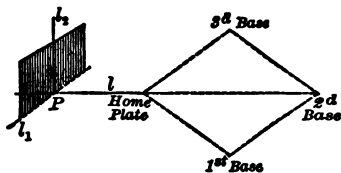
3. If a plane is perpendicular to a line-segment PP' at its middle point, prove: (1) Every point in the plane is equally distant from P and P'; (2) every point equally distant from P and P' lies in this plane. What is the locus of all points in space equidistant from P and P'? Compare § 47.

4. Given the points A and B not in a plane M. Find the locus of all points in M equidistant from A and B.

Suggestion. All such points must lie in the plane M and also in the plane which is the perpendicular bisector of the segment AB.

5. Find the locus of all points equidistant from two given points A and B, and also equidistant from two points C and D. Discuss.

6. State and prove a theorem of solid geometry corresponding to the theorem of plane geometry given in § 38.
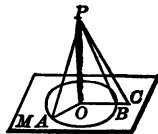
7. If in the figure $PD \perp$ plane M, and $DC \perp AB$, a line of the plane M, prove that $PC \perp AB$.
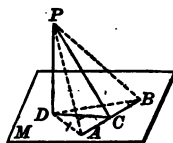
Suggestion. Lay off $CA = CB$, and compare triangles.

8. If in the same figure $PD \perp M$, and $PC \perp AB$, a line of the plane, prove that $DC \perp AB$.

PARALLEL LINES PERPENDICULAR TO THE SAME PLANE

83. THEOREM VII.　*Two lines perpendicular to the same plane are parallel; and*

Conversely, if one of two parallel lines is perpendicular to a plane, the other is also.
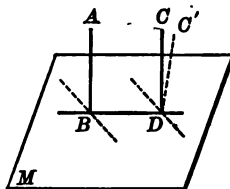
FIG. 1.　　　　　　FIG. 2.

Given (1) *AB* and *CD* each ⊥ the plane *M*.　Fig. 1.

To prove that　　　　*AB* ∥ *CD*.

Proof:　　Draw *BD* and make *DE* ⊥ *DB*.

Take points *A* and *E* so that *BA = DE*, and draw *AD*, *AE*, and *BE*.

Now prove: (1) △ *ABD* = △ *BDE* and ∴ *AD = BE*;

(2) △ *ADE* = △ *ABE* and ∴ ∠ *ADE* = ∠ *ABE* = Rt. ∠.

∴ *DC*, *DA*, *DB*, and *BA* all lie in the same plane.　§ 79

∴ *AB* ∥ *CD*.　　　　　　§ 34

Given (2) *AB* ∥ *CD* and *AB* ⊥ *M*.　Fig. 2.

To prove that　　　　*CD* ⊥ *M*.

Proof:　　If *CD* is not ⊥ *M* let *C'D* be ⊥ *M*.

Then *C'D* ∥ *AB* by case (1), and *C'D* coincides with *CD* § 21

and　　　　　∴ *CD* ⊥ *M*.

84. COROLLARY 1.　*If each of two lines is parallel to a third line they are parallel to each other.*

85. COROLLARY 2.　*If a plane is perpendicular to one of two parallel lines, it is perpendicular to the other.*
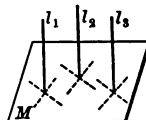
PARALLEL PLANES AND LINES

86. Parallel Planes. Two planes which do not meet are said to be *parallel*.

87. Line Parallel to a Plane. A straight line and a plane which do not meet are said to be parallel.

88. Intercepted Segments. If a straight line l_2 meets two planes in A and B, then the segment AB is said to be *intercepted* by the planes.
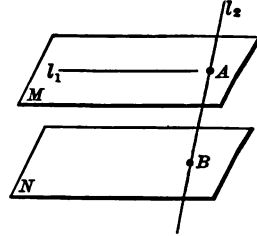
Any line, as l_1, in either of two parallel planes, M and N, is parallel to the other plane. The segment AB on the line l_2 is intercepted by the planes.

LINE PARALLEL TO A PLANE

89. THEOREM VIII. *If a straight line is parallel to a given plane, it is parallel to the intersection of any plane through it with the given plane.*
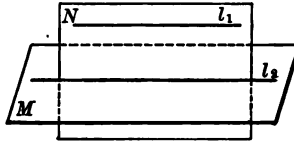
Suggestion for proof. If l_1 is the given line, M the given plane, and l_2 the intersection of a plane N through l_1 with M, show that l_1 and l_2 lie in plane N and cannot meet.

90. COROLLARY 1. *If a line outside a plane is parallel to some line in the plane, then the first line is parallel to the plane.*

91. COROLLARY 2. *If a line is parallel to a plane, then through any point in the plane there is a line in the plane parallel to the given line.*

92. COROLLARY 3. *The intersections of a plane with two parallel planes are parallel lines.*
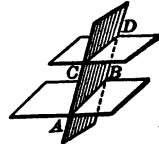
PLANES PERPENDICULAR TO A LINE ARE PARALLEL

93. THEOREM IX. *If each of two planes is perpendicular to the same line, they are parallel; and*

Conversely, if one of two parallel planes is perpendicular to a line, the other is also.
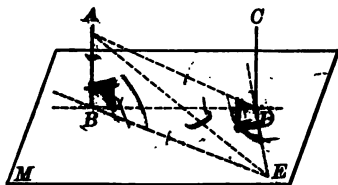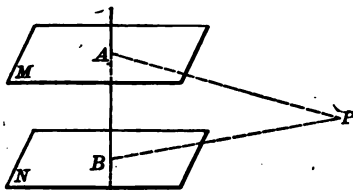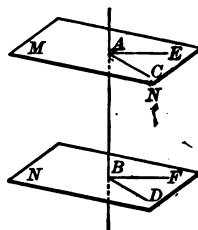

FIG. 1.   FIG. 2.

Given (1) plane $M \perp AB$ and plane $N \perp AB$.   Fig. 1.

To prove that $M \parallel N$.

Proof: Suppose M and N to meet in some point P.   Draw AP in M and BP in N.   Then $AB \perp AP$ and $AB \perp BP$ (§ 71), which is impossible.   § 29

Given (2) $M \parallel N$ and $M \perp AB$.   Fig. 2.

To prove that $N \perp AB$.

Proof: Through AB pass a plane cutting M and N in AC and BD, and a second plane cutting M and N in AE and BF.

Then, $AC \parallel BD$ and $AE \parallel BF$.   § 92
Now prove   (1) $AB \perp BD$, (2) $AB \perp BF$.   § 35
                    $\therefore AB \perp N$.   § 76

94. COROLLARY 1.   *Parallel line-segments included between parallel planes are equal.*

95. COROLLARY 2.   *If a line is perpendicular to one of two parallel planes, it is perpendicular to the other also.*

96. COROLLARY 3.   *Two planes each parallel to a third plane are parallel to each other.*

PARALLEL PLANES

97. **THEOREM** X. *If a plane is parallel to each of two intersecting lines, it is parallel to the plane of these lines.*
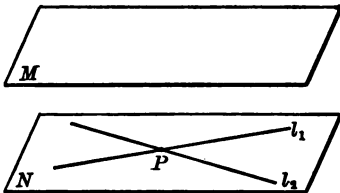
Given a plane *M* parallel to the intersecting lines l_1 and l_2.

To prove that the plane M is ∥ the plane N of l_1 and l_2.

Proof: If *M* is not ∥ *N*, these planes meet in a line l_3.   § 69

Then neither l_1 nor l_2 can meet l_3, since they are ∥ *M*.

Hence $l_1 ∥ l_3$, and $l_2 ∥ l_3$, which is impossible.   § 21

∴ *M* and *N* cannot meet and are parallel.

Q. E. D.

98. **THEOREM** XI. *Through a point not in a plane there is one and only one plane parallel to this plane.*

Given a plane *M* and a point *P* not in *M*.

To prove that through P there is one and only one plane ∥ M.
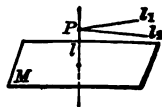
Proof: Let *l* be a line through *P* and ⊥ *M*.

Through *P* draw l_1 and l_2 each ⊥ *l*.   § 75

The plane *N* of l_1 and l_2 is ⊥ *l*, and hence *N* ∥ *M*.   §§ 76, 93

If through *P* there were another plane *R* ∥ *M*, then

R would be ⊥ *l* at *P*.   § 93

But *N* and *R* cannot both be ⊥ *l* at *P*.   § 78

Hence *N* is the only plane through *P* ∥ *M*.

Q. E. D.

PLANES PARALLEL TO GIVEN LINES

99. Theorem XII. *Through one of two skew lines there is one and only one plane parallel to the other line.*
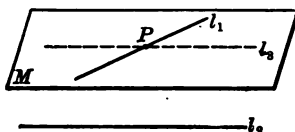
Given two skew lines l_1 and l_2. See § 7.

To prove that through l_1 there is one and only one plane ∥ l_2.

Proof: Through P, a point in l_1, draw a line l_3 ∥ l_2.

Then l_1 and l_3 determine a plane M ∥ l_2.   § 90

Any other plane N through l_1 would meet the plane of l_3 and l_2 in a line through P not ∥ l_2 and hence N would meet l_2.

<div align="right">Q. E. D.</div>

100. Theorem XIII. *Through a point outside of each of two non-parallel lines there is one and only one plane parallel to both of these lines.*
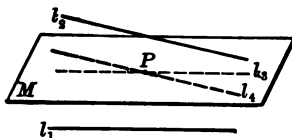
Given a point P outside of the non-parallel lines l_1 and l_2.

To prove that there is one and only one plane M through P parallel to l_1 and l_2.

Proof: Through P pass l_3 ∥ l_1 and l_4 ∥ l_2.

Then the plane M of l_3 and l_4 is parallel to l_1 and l_2.   § 90

In any other plane N through P ∥ l_1 and l_2, there are lines l'_3 and l'_4 through P such that l'_3 ∥ l_1 and l'_4 ∥ l_2.   § 91

In that case l'_3 is identical with l_3 and l'_4 with l_4.   § 21

Hence any such plane N is identical with M.   Q. E. D.

EXERCISES

1. Given a plane M and a point P not in M. Find the locus of the middle points of all segments connecting P with points in M.

Suggestion. Use the fact that a line parallel to the base of a triangle and bisecting one side bisects the other side also.

2. Show that a plane containing one only of two parallel lines is parallel to the other.

3. If in two intersecting planes a line of one is parallel to a line of the other, then each of these lines is parallel to the line of intersection of the planes.

4. Show that three lines which do not meet in one point must all lie in the same plane if each intersects the other two.

5. Show that three planes, each of which intersects the other two, have a point in common unless their three lines of intersection are parallel.

Suggestion. Suppose two of the intersection lines are not parallel, but meet in some point O. Then show that the other line of intersection passes through O, and hence that O is the point common to all three planes.

6. Given two intersecting planes M and N. Find the locus of all points in M at a given perpendicular distance from N.

7. Given two non-intersecting lines l_1 and l_2. Find the locus of all lines meeting l_1 and parallel to l_2.

8. Prove that the middle points of the sides of any quadrilateral in space are the vertices of a parallelogram.

Suggestion. Use the fact that a line bisecting two sides of a triangle is parallel to the third side. Note that the four vertices of a quadrilateral in space do not necessarily all lie in the same plane.

State the corresponding theorem in plane geometry.

9. In erecting a flagpole on a level space, show how it can be made perpendicular by means of three ropes of equal length. See § 81.

ANGLES WHOSE SIDES ARE PARALLEL

101. THEOREM XIV. *If two intersecting lines in one plane are parallel, respectively, to two intersecting lines in another plane, then the two planes are parallel, and the corresponding angles formed by the lines are equal.*
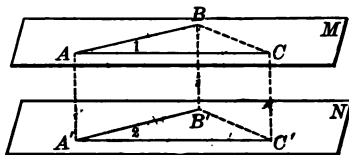
Given the planes M and N in which $AB \parallel A'B'$, and $AC \parallel A'C'$.

To prove that $M \parallel N$ and $\angle 1 = \angle 2$.

Proof: (1) If M is not $\parallel N$, then these planes meet in a line l. Why?

Then neither AB nor AC can meet l since each is $\parallel N$.

$\therefore AB \parallel l$ and $AC \parallel l$ which is impossible. Why?

(2) To prove $\angle 1 = \angle 2$, lay off $AB = A'B'$, $AC = A'C'$, and draw BC, $B'C'$, AA', BB', and CC'.

Analysis: $\angle 1 = \angle 2$ if $\triangle ABC = \triangle A'B'C'$, which is true if $BC = B'C'$.

But $BC = B'C'$ if $BB'C'C$ is a $\square$, which is so if $BB' = CC'$ and $BB' \parallel CC'$. This last is true if $AA'C'C$ and $AA'B'B$ are $\square$, for then $BB' = AA' = CC'$ and $BB' \parallel AA' \parallel CC'$.

Hence we need to prove in order (1) $AA'C'C$ and $AA'B'B$ are $\square$, (2) $BB'C'C$ is a $\square$, (3) $\triangle ABC = \triangle A'B'C'$, and (4) $\angle BAC = \angle B'A'C'$.

Hence it is proved that the planes are parallel and the angles are equal. Q. E. D.

102. COROLLARY. *Two angles in space whose sides are parallel each to each are either equal or supplementary.*

PARALLEL PLANES INTERCEPT PROPORTIONAL SEGMENTS

103. THEOREM XV. *If two straight lines are cut by three parallel planes, the intercepted segments on one line are proportional to the corresponding segments on the other.*
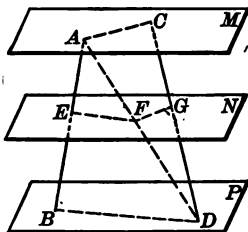
Given the lines *AB* **and** *CD* **cut by the planes** *M, N,* **and P.**

To prove that $\dfrac{AE}{EB} = \dfrac{CG}{GD}$.

Outline of Proof: Draw *AD* and let the plane determined by *AD* and *CD* cut the planes *M* and *N* in *AC* and *FG* respectively; and let the plane of *AB* and *AD* cut *N* and *P* in *EF* and *BD* respectively.

Then prove (1) *FG* ∥ *AC* and *EF* ∥ *BD*,

(2) $\dfrac{AE}{EB} = \dfrac{AF}{FD}$ and $\dfrac{CG}{GD} = \dfrac{AF}{FD}$, (3) $\dfrac{AE}{EB} = \dfrac{CG}{GD}$.

104. COROLLARY. *Parallel planes which intercept equal segments on any transversal line intercept equal segments on every transversal line.*

SIGHT WORK

Review the theorems of solid geometry up to this point by stating the corresponding theorems of plane geometry, in case such theorems exist.

EXERCISES

1. If the spaces between four parallel shelves are 5, 8, and 10 inches respectively, and a slanting rod intersecting them has a 7-inch segment between the first two shelves, find the other two segments of the rod.

2. If a line cuts three parallel planes, M, N, R, so that the segment intercepted between M and N is 7 and that between N and R is 21, and if another line cuts the same planes so that the segment between M and N is 11, find the segment on the second line between N and R.

3. Show that line-segments included between parallel planes and perpendicular to them are equal, and hence that parallel planes are everywhere equally distant. How can a carpenter make use of this principle in placing two parallel shelves? How many distances must he measure? Why?

4. Show that through a point outside a plane any number of lines can be drawn parallel to the plane. How are all these parallels related?

5. Prove that if a plane bisects two sides of a triangle it is parallel to the third side.

6. The perpendicular distance from a point P to a plane is 12 in. Find the radius of the circle which is the locus of all points in the plane at a distance of 20 in. from P.

7. Show that, if three line-segments not in the same plane are equal and parallel, the triangles formed by joining their extremities, as in the figure of § 101, are equal and their planes are parallel.

8. What is the relation of two lines if they are (a) parallel to a given line, (b) perpendicular to a given line, (c) parallel to a given plane, (d) perpendicular to a given plane?

9. What is the relation of two planes if they are both (a) parallel to a given plane, (b) parallel to a given line, (c) perpendicular to a given line?

DIHEDRAL ANGLES

105. Dihedral Angle. The part of a plane on one side of a line in it is called a *half-plane*. The line is called the *edge* of the half-plane. Two half-planes meeting in a common edge form a *dihedral angle*. The common edge is the *edge* of the angle and the half-planes are its *faces*.

106. Plane Angle of a Dihedral Angle. Two lines in the respective faces of a dihedral angle and perpendicular to its edge at a common point form a plane angle, which is called *the plane angle of the dihedral angle*.

In the figure, the half-planes M and N form the dihedral angle M-AB-N, read by naming the two faces and the edge. If CD in N is $\perp AB$ and ED in M is $\perp AB$, then $\angle CDE$ is the plane angle of the dihedral angle M-AB-N.
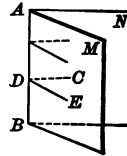
By § 101 all plane angles of a dihedral angle are equal to each other.

107. Generation of a Dihedral Angle. A dihedral angle may be thought of as *generated* by the rotation of a half-plane about its edge. The magnitude of the angle depends solely upon the *amount of rotation*.
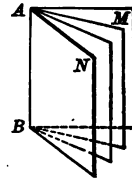
108. Equal Dihedral Angles. Two dihedral angles are *equal* when they can be so placed that they coincide.

109. Right Dihedral Angle. A right dihedral angle is one whose *plane angle* is a right angle.
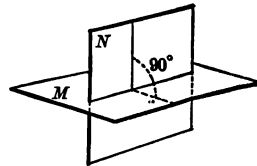
110. Perpendicular Planes. Two planes are said to be *mutually perpendicular* if their dihedral angle is a right angle.

Dihedral angles are *acute* or *obtuse* according as their plane angles are acute or obtuse.

Two dihedral angles are *adjacent, vertical, supplementary,* or *complementary* according as their plane angles possess these properties.

DIHEDRAL ANGLES AND THEIR PLANE ANGLES

111. THEOREM XVI. *Two dihedral angles are equal if their plane angles are equal; and*

Conversely, if two dihedral angles are equal, their plane angles are equal.
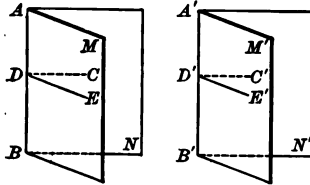
Given (1) the dihedral angles *M–AB–N* and *M'–A'B'–N'* in which the plane ∠s *CDE* and *C'D'E'* are equal.

To prove that ∠ *M–AB–N* = ∠ *M'–A'B'–N'*.

Proof: Place the equal ∠s *CDE* and *C'D'E'* in coincidence.

Then *AB* coincides with *A'B'* since these lines are perpendicular to the plane *CDE* at the point *D*.   § 80

Then *M* and *M'* coincide as do also *N* and *N'* since they are determined by coincident lines.   § 73

∴ ∠ *M–AB–N* = ∠ *M'–A'B'–N'*.   § 108

Given (2) ∠ *M–AB–N* = ∠ *M'–A'B'–N'*.

To prove that the plane ∠s *CDE and C'D'E' are equal.*

Proof: Place the equal dihedral ∠s *M–AB–N* and *M'–A'B'–N* in coincidence so that the points *D* and *D'* coincide.

Then *CD* and *C'D'* coincide as do also *ED* and *E'D'*.   § 29

∴ ∠ *CDE* = ∠ *C'D'E'*.   Q. E. D.

112. COROLLARY. *All right dihedral angles are equal.*

113. Measure of a Dihedral Angle. It follows from § 111 that the plane angle of a dihedral angle may be regarded as its measure.

MUTUALLY PERPENDICULAR PLANES

114. THEOREM XVII. *If two planes are mutually perpendicular, and if a line is drawn from a point in one perpendicular to their intersection, then this line is perpendicular to the second plane.*
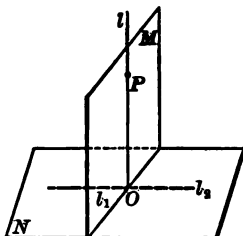
Given plane $M \perp$ plane N and a point P in M. Let l_1 be the intersection of M and N and l a line in M through $P \perp l_1$.

To prove that $\qquad l \perp N$.

Proof: Let l meet l_1 in O. Through O draw l_2 in N and perpendicular to l_1.

Then the angle formed by the lines l and l_2 is the plane angle of the dihedral angle between the planes M and N.   § 106

$$\therefore l \perp l_2 \qquad\qquad\qquad \text{§ 109}$$

$$\therefore l \perp N. \qquad\qquad\qquad \text{Why?}$$

Q. E. D.

115. COROLLARY. *If two planes are mutually perpendicular and if a line is drawn from a point in one and perpendicular to the other, then this line lies in the first plane and is perpendicular to the intersection of the two planes.*

Proof: By the theorem there is a line through P which lies in the plane M and is $\perp N$. But through P there is only one line $\perp N$.   § 80

PLANE PERPENDICULAR TO EACH OF TWO PLANES

116. Theorem XVIII. *If a plane is perpendicular to each of two planes, it is perpendicular to their line of intersection.*
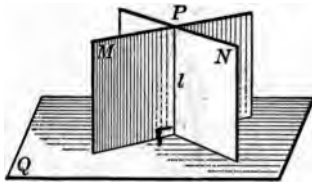
Given $Q \perp M$ and $Q \perp N$ and l the intersection of M and N.

To prove that $Q \perp l.$

Proof: From a point P common to M and N draw a line $l' \perp Q$.

Then l' lies in both M and N. § 115

Hence, l' and l are the same line. Why?

That is, $l \perp Q$, or $Q \perp l.$ Q. E. D.

SIGHT WORK

1. State theorems on dihedral angles corresponding to those in §§ 24, 30, 31, 32, on plane angles.

2. Name all dihedral angles in the accompanying figure.

3. Any plane $\perp$ the edge of a dihedral angle is $\perp$ each of its faces. Why?

4. If each of three lines is $\perp$ the other two at the same point, then each is $\perp$ the plane of the other two. Why?
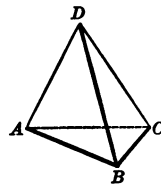
5. Find the locus of all points at a given distance from a given plane and also at a given distance from a second plane. Discuss this locus for the various cases possible.

PERPENDICULAR LINES AND PLANES

117. THEOREM XIX. *If a line is perpendicular to a plane, every plane containing this line is perpendicular to that plane.*
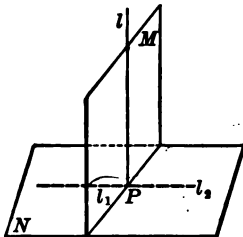
Given a line $l \perp$ plane N at P.

To prove that a plane M containing l is $\perp N$.

Proof : Let l_1 be the intersection of M and N. In N draw $l_2 \perp l_1$ at P.

Then $l \perp l_1,\ \ l_2 \perp l_1,$ and $l \perp l_2.$ **Why ?**

                    $\therefore M \perp N.$ **Why ?**

<div align="right">Q. E. D.</div>

118. Projection of a Figure on a Plane. The projection of a *point* on a plane is the foot of the perpendicular from the point to the plane. The projection of any *figure* on a plane is the locus of the projections of all points of the figure on the plane.

Thus, B is the projection of the point A on M, and l_2 is the projection of the line l_1 on M.
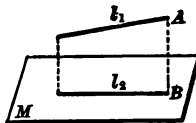
119. Angle Between Line and Plane. The angle between a plane and a line oblique to it is understood to be the *acute angle* formed by the line and its projection upon the plane.

120. Bisector of a Dihedral Angle. A half-plane bisects a dihedral angle if it passes through the edge of the dihedral angle and bisects its plane angle.

PROJECTION OF A STRAIGHT LINE ON A PLANE

121. THEOREM XX.   *The projection of a straight line on a plane is a straight line in that plane.*
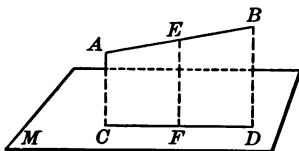
Given a plane *M* and a line *AB* projected upon it.

To prove that the projection is a straight line.

Proof: Project any three points A, E, B of the line AB into the points C, F, D in the plane M.

Then AC and BD determine a plane $\perp M$.   §§ 83, 117

Hence, by §§ 115, 69, any three points, C, F, D, of the projection lie in a straight line, that is, the projection is a straight line.   Q. E. D.

122. THEOREM XXI.   *The acute angle formed by a straight line with its own projection on a plane is the least angle which it makes with any line in that plane.*
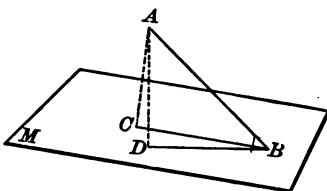
Given *CB* the projection of *AB* on the plane *M*, and any other line *BD* in *M* through *B*.

To prove that $\angle ABD > \angle ABC$.

Proof: Draw $AC \perp M$, make $BD = BC$, and draw AD.

Then prove $AD > AC$ and $\therefore \angle ABD > \angle ABC$.   § 33

PLANE BISECTING A DIHEDRAL ANGLE

123. THEOREM XXII.   *The locus of all points equally distant from the faces of a dihedral angle is the half-plane bisecting the angle.*
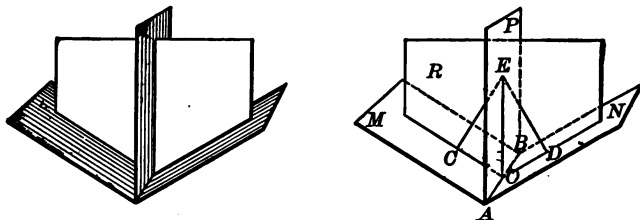
Given the half-plane P bisecting the dihedral $\angle$ *M–AB–N*.

To prove (1) *that any point E in P is equally distant from M and N, and* (2) *that any point which is equally distant from M and N lies in P.*

Proof: (1)   Draw $EC \perp M$ and $ED \perp N$.

Then (*a*) Plane $CED \perp M$ and also $\perp N$.   **Why?**

   (*b*) Plane $CED \perp AB$.   **Why?**

Now $\angle COE$ is the plane $\angle$ of M–AB–P,
and $\angle DOE$ is the plane $\angle$ of N–AB–P.   **Why?**

$\therefore \angle COE = \angle DOE$ by the hypothesis.

Hence $EC = ED$.   §§ 42, 23

(2) Let E be any point such that $EC = ED$, these being perpendicular respectively to M and N.

Then $\angle COE = \angle DOE$.   §§ 40, 23

Hence, the plane P determined by ABE is the bisector of the given dihedral angle, that is, the point E lies in P.   q. e. d.

SIGHT WORK

Define a locus.   Why are two conditions necessary in determining a locus?   Are these conditions fulfilled in the above theorem?

PROBLEMS IN CONSTRUCTION

In making constructions in solid geometry, it is assumed that

124. *A line can be passed through any two points.*

125. *A plane can be passed through three non-collinear points.*

126. *Through a point not on a given line there can be drawn a line parallel to this line.*

127. *Through a point a line perpendicular to a given line can be drawn.*

128. COROLLARY. *A plane can be passed through* (1) *a line and a point not on it,* (2) *two intersecting lines,* (3) *two parallel lines.* § 73

Q. E. F.

CONSTRUCTING A PLANE PERPENDICULAR TO A LINE

129. PROBLEM. *Through a given point to construct a plane perpendicular to a given line.*
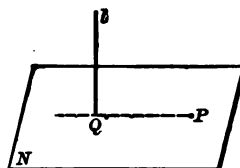
FIG. 1.   FIG. 2.

(1) Given a line *l* and a point *P* on it. Fig. 1.

To construct a plane M ⊥ l at P.

Construction: Through *P* draw two lines *PA* and *PB* each perpendicular to *l*. §§ 127, 75

Then the plane *M* determined by *PA* and *PB* is ⊥ *l*. § 76

(2) Given a line *l* and a point *P* not on it. Fig. 2.

To construct a plane N through P and ⊥ l.

Construction: From *P* draw a line *PQ* ⊥ *l*. § 127

At *Q* construct a plane *N* ⊥ *l*, as in (1).

Then, by § 79, *PQ* lies in *N*, and hence *N* is the plane required. Q. E. F.

130. PROBLEM.   *Through a given point to construct a line perpendicular to a given plane.*
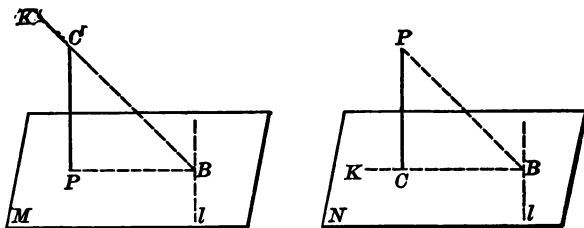
(1) **Given a plane *M* and a point *P* in it.**

To construct a line through P and ⊥ M.

Construction: Let *l* be any line in *M* not passing through *P*. Draw $PB \perp l$.   Through *B* draw $BK \perp l$ but not in *M*. In the plane *PBK* draw $PC \perp PB$ and meeting *BK* in *C*.

Prove $PC \perp M$.   §§ 76, 117, 114

(2) **Given a plane *N* and a point *P* not in it.**

To construct a line through P and ⊥ N.

Construction: Let *l* be any line in *N*.   Draw $PB \perp l$ and in *N* draw $BK \perp l$.   In the plane *PBK* draw $PC \perp BK$.

Prove, as in (1), $PC \perp N$.   (See also § 80.)

Q. E. F.

131. PROBLEM.   *Through one of two skew lines to pass a plane parallel to the other.*

Given two skew lines l_1 and l_2.

To construct a plane through $l_1 \parallel l_2$.
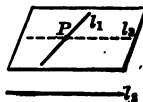
Construction: Through a point *P* in l_1 pass a line $l_3 \parallel l_2$.   (§ 126.)

The plane determined by l_1 and l_3 is $\parallel l_2$.   § 90

Q. E. F.

CONSTRUCTING A PLANE PERPENDICULAR TO A PLANE

132. PROBLEM. *Through a given line to pass a plane perpendicular to a given plane.*
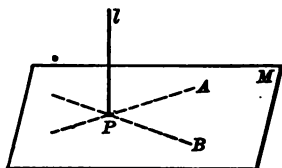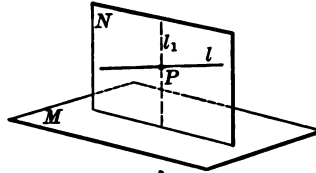

Given a line *l* and a plane *M*.

To pass a plane through l and ⊥ *M.*

Construction: Through a point *P* in *l* pass a line l_1 ⊥ *M*.

§ 130

The plane *N* determined by *l* and l_1 passes through *l* and is ⊥ *M*.

§ 117

Q. E. F.

CONSTRUCTING A PLANE PARALLEL TO EACH OF TWO LINES

133. PROBLEM. *Through a point to pass a plane parallel to each of two given lines.*

Given a point *P* and two lines l_1 and l_2.

To construct a plane through P parallel to l_1 and l_2.

Construction: Through *P* pass lines l'_1 and l'_2 such that $l'_1 \parallel l_1$ and $l'_2 \parallel l_2$.

§ 126

Then the plane determined by l'_1 and l'_2 is parallel to both l_1 and l_2.

§ 90

Q. E. F.

SIGHT WORK

1. Study the special case of § 132 in which the given line is perpendicular to the given plane.

2. Is it possible that the plane constructed by the method of § 133 may contain one or both of the given lines?   Study all possible cases.

3. Through a given point pass a plane ⊥ each of two given planes. How many such can be constructed?   Study the case in which the two given planes are parallel to each other.

Suggestion.   Consider the relation of the required plane to the intersection of the two given planes.

134. PROBLEM. *To construct a common perpendicular to two skew lines.*
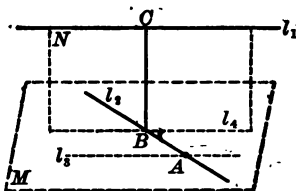
Given two skew lines l_1 and l_2.

To construct a line BC perpendicular to each of them.

Construction: Through A, any point in l_2, draw $l_3 \parallel l_1$.
Let M be the plane determined by l_2 and l_3.
Through l_1 pass a plane $N \perp M$ and meeting M in l_4.
At B the intersection of l_2 and l_4 erect $BC \perp M$.
Then BC is the required line perpendicular to l_1 and l_2.

Outline of Proof: Show each of the following:
(1) BC lies in the plane N, and $\therefore$ meets l_1.
(2) $l_1 \parallel l_4$, (3) $BC \perp l_4$, and $\therefore$ $BC \perp l_1$.

135. COROLLARY. *There cannot be more than one common perpendicular to two skew lines.*

Suggestion. Suppose a second common perpendicular drawn from a point P_1 in l_1 to a point P_2 in l_2. Then show that from P_1 it would be possible to have two lines $\perp$ plane M.

<div align="center">

SIGHT WORK

</div>

1. If a line is $\perp$ a plane, show that its projection is a point.

2. If a line-segment is $\parallel$ a plane, show that its projection is a segment equal to the given segment.

3. If a line-segment is oblique to a plane, show that its projection is less than the given segment.

EXERCISES

1. Show that the projections of two parallel lines on a plane are parallel. Is the converse true? Illustrate with pieces of cardboard.

2. If two parallel lines meet a plane, they make equal angles with it. Why? Is the converse true? What is the corresponding theorem in plane geometry? Is its converse true?

3. If a line cuts two parallel planes, it makes equal angles with them. Why? Is the converse true? Discuss the corresponding theorems in plane geometry.

4. If two parallel line-segments are oblique to a plane, their projections on the plane are in the same ratio as the given segments.

5. A line and its projection on a plane determine a plane perpendicular to the given plane.

6. If a line is parallel to one of two given planes and perpendicular to the other, then the two planes are perpendicular to each other.

7. Find the locus of all points equidistant from two points A and B and also at a given distance from a plane M. Discuss.

8. Find the locus of all points equidistant from two given planes and also equidistant from two given points. Discuss.

9. Find the locus of all points equidistant from two given planes M and N, and also equidistant from two other given planes Q and R. Discuss. Compare with the corresponding loci in plane geometry.

10. Prove that there is a line in space every point of which is equidistant from three points A, B, C, provided these points do not lie on one line.

POLYHEDRAL ANGLES

136. Polyhedral Angle. Given a convex polygon and a point *P* not in its plane. If a half-line *l* with its end point fixed at *P* moves so that it always touches the polygon and is made to traverse it completely, it is said to generate a *convex polyhedral angle.*

137. Vertex. Edges. The fixed point is the *vertex* of the polyhedral angle, and the rays through the vertices of the polygon are the *edges* of the polyhedral angle.

Any two consecutive edges determine a plane, and the portion of such a plane included between these edges is called a *face* of the polyhedral angle.
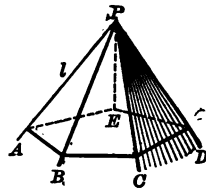
138. Face Angles. The plane angles at the vertex are called the *face angles* of the polyhedral angle. A polyhedral angle having three faces is called a *trihedral angle.*

Thus in the figure, *P* is the vertex, *PA*, *PB*, etc., are the edges, and ∠*APB*, ∠*BPC*, etc., are the face angles.

139. Equal Polyhedral Angles. Two polyhedral angles are *equal* if they can be made to coincide.

A polyhedral angle is read by naming the vertex and one letter in each edge, as *P-ABCDE*, or by naming the vertex alone where no ambiguity would arise.

140. Order of Parts in Triangles. In the triangles *ABC,* and *A'B'C'*, *AB = A'B'*, *BC = B'C'*, *CA = C'A'.* However, the sides *AB*, *BC*, *CA*, are arranged

in counter-clockwise order, and the sides *A B*, *B C'*, *C'A'*, are arranged in clockwise order. That is, the parts of the two triangles are arranged in opposite orders.
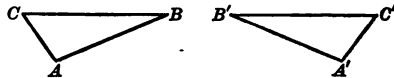
141. Order of Parts in Polyhedral Angles. In the trihedral angles O and O' the face angles AOB, BOC, COA are arranged in *counter-clockwise* order as viewed from the vertex, while $A'O'B'$, $B'O'C'$, $C'O'A'$ are arranged in *clockwise* order.
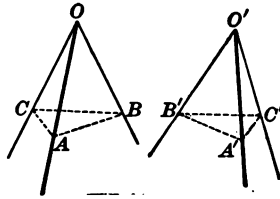
The trihedral angles O and O' cannot be made to coincide, even though their corresponding parts are equal. This can be illustrated by attempting to put a left glove on the right hand.

CONDITIONS WHICH MAKE TRIHEDRAL ANGLES EQUAL

142. THEOREM XXIII. *Two trihedral angles are equal if two face angles and the included dihedral angle of one are equal respectively to two face angles and the included dihedral angle of the other and arranged in the same order.*
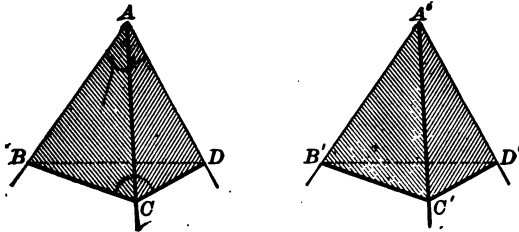
Suggestion for Proof: This and the following theorem may be proved by superposition in the same manner as the corresponding theorems on the equality of triangles in plane geometry.

143. THEOREM XXIV. *Two trihedral angles are equal if a face angle and the two adjacent dihedral angles of one are equal respectively to a face angle and the adjacent dihedral angles of the other and arranged in the same order.*

EQUAL DIHEDRAL ANGLES OPPOSITE EQUAL FACE ANGLES

144. THEOREM XXV. *If two trihedral angles have the three face angles of the one equal respectively to the three face angles of the other, the dihedral angles opposite the equal face angles are equal.*
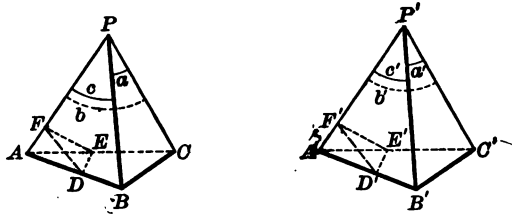
Given the trihedral angles P and P', in which $\angle a = \angle a'$, $\angle b = \angle b'$, $\angle c = \angle c'$.

To prove that the corresponding dihedral angles are equal.

Outline of Proof: Let P and P' be cut by the planes ABC and $A'B'C'$, making $PA = PB = PC = P'A' = P'B' = P'C'$.

On the edges PA and $P'A'$ lay off $AF = A'F''$, and through F and F'' pass planes $\perp$ to PA and $P'A'$ respectively.

AF and $A'F'$ are to be taken short enough so that these planes shall cut the segments AB, AC and $A'B'$, $A'C'$ respectively.

In order to prove that $\angle B\text{–}AP\text{–}C = \angle B'\text{–}A'P'\text{–}C'$, *i.e.* that $\angle DFE = \angle D'F'E'$, use the pairs of $\triangle\!\!\!\triangle$ APB, $A'P'B'$; BPC, $B'P'C'$; CPA, $C'P'A'$; ABC, $A'B'C'$. Then prove in order:

    (1) $\triangle ADF = \triangle A'D'F'$;   (2) $\triangle AEF = \triangle A'E'F'$;

    (3) $\triangle ADE = \triangle A'D'E'$;   (4) $\triangle DEF = \triangle D'E'F'$.

$$\therefore \angle DFE = \angle D'F'E'.$$

145. COROLLARY. *Two trihedral angles are equal if the face angles of one are equal respectively to the face angles of the other and arranged in the same order.*

SYMMETRICAL TRIHEDRAL ANGLES

146. Symmetrical Trihedral Angles. Two trihedral angles are *symmetrical*, one to the other, if the face angles and the dihedral angles of one are equal respectively to the face angles and the dihedral angles of the other, but arranged in the *opposite order*.

147. Vertical Trihedral Angles. Two trihedral angles are *vertical* if the edges of one are the prolongations of the edges of the other.
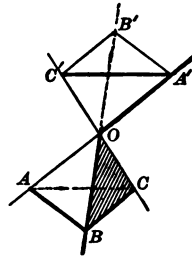
148. COROLLARY 1. *Two vertical trihedral angles are symmetrical.*

The proof is evident from the figure above.

149. COROLLARY 2. *Two trihedral angles are symmetrical if the face angles of one are equal to the face angles of the other and arranged in the opposite order.*

SIGHT WORK

1. Is it possible that all three face angles of a trihedral angle shall be right angles ? Is it possible that all three dihedral angles shall be right angles ?

Suggestion. Consider a corner of a rectangular box.

2. If three planes meet in a point how many trihedral angles are formed ? If all face angles of one of these trihedral angles are right angles, what about the face angles of the other trihedral angles ?

3. If in the figure of § 144 the $\angle a = \angle b = \angle c = 60°$, what can be said about the triangles *PAB*, *PBC*, *PCA*, and *ABC* ?

4. Show how to locate a point which is at a distance of 2 feet from each of the three planes determined by the faces of a trihedral angle. How many such points are there ?

Suggestion. Pass planes parallel to each face of the trihedral angle, one on each side of it, and at a distance of 2 feet from it.

CONDITIONS WHICH MAKE TRIHEDRAL ANGLES SYMMETRICAL

150. THEOREM XXVI. *Two trihedral angles are symmetrical*

(1) *if two face angles and the included dihedral angle of one are equal respectively to two face angles and the included dihedral angle of the other, but arranged in the opposite order; or*

(2) *if a face angle and the adjacent dihedral angles of one are equal respectively to a face angle and the adjacent dihedral angles of the other, but arranged in the opposite order.*
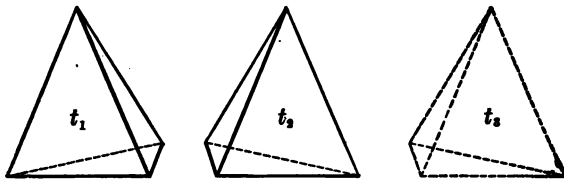
Proof: Let t_1 and t_2 be two trihedral angles having the properties specified under (1). Let t_3 be a trihedral angle symmetrical to t_1. Then $t_2 = t_3$. § 142

∴ t_1 and t_2 are symmetrical.

The proof of the second part is left to the student.

151. THEOREM XXVII. *The sum of two face angles of a trihedral angle is greater than the third face angle.*

Outline of Proof: Connect points A and C on two sides. Suppose $\angle ADC > \angle ADB$. Construct DE making $\angle ADE = \angle ADB$. Suppose that point B is taken so that $DB = DE$, and that AB and BC are drawn. Then prove

(1) $AB = AE$, (2) $AC < AB + BC$,
(3) $EC < BC$, (4) $\angle EDC < \angle BDC$,
(5) $\angle ADC < \angle ADB + \angle BDC$.
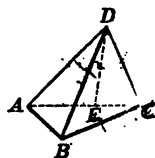
SUM OF THE FACE ANGLES OF A POLYHEDRAL ANGLE

152. THEOREM XXVIII. *The sum of the face angles of any convex polyhedral angle is less than four right angles.*
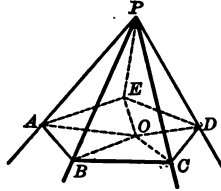
Proof: Let *ABCDE* be a plane section of the given polyhedral angle. The number of triangles thus formed having *P* for a vertex is equal to the number of face angles of the polyhedral angle.

Let *O* be any point in the base, and draw *OA*, *OB*, *OC*, etc.

Then $\angle PBA + \angle PBC > \angle ABC$, $\angle PCB + \angle PCD > \angle BCD$, and so on.   § 151

Now the sum of the ∠s of the △ *OAB, OBC*, etc., is equal to the sum of the ∠s of the △ *PAB, PBC*, etc.

Hence, $\angle APB + \angle BPC + \cdots < \angle AOB + \angle BOC + \cdots$.

But the sum of the ∠s about *O* is four right angles.

Therefore, the sum of the face angles of the polyhedral angle is less than four right angles.   Q. E. D.

Note. If as in § 144 the three edges of a trihedral angle are cut by a plane, the intersection is a triangle. Then the sides and angles of such a triangle *correspond* to the face angles and dihedral angles respectively of the trihedral angle.

SIGHT WORK

1. What theorem in plane geometry corresponds to the theorem of § 151 in the sense of the above note ?

2. Discuss propositions of plane geometry corresponding to those of § 150

SUMMARY OF BOOK I

1. State the axioms used in this Book.

2. Describe the various ways of determining a plane.

3. State the definitions on perpendicular lines and planes.

4. State a theorem on a line through a point perpendicular to a plane.

5. State a theorem on a plane through a point perpendicular to a line.

6. State the definitions on parallel lines and planes.

7. State propositions on (1) a plane through a point parallel to a given plane; (2) a plane through a line parallel to a given line; (3) a plane through a point parallel to two given lines.

8. State some facts about perpendiculars in the plane which do not hold in space.

9. Define a dihedral angle and the plane angle of a dihedral angle.

10. What theorems on perpendicular planes are proved in connection with dihedral angles? Notice that these theorems include all those on one plane perpendicular to another. Why should this be the case?

11. State the definitions and theorems on projections given in this Book.

12. State the definitions and theorems on polyhedral angles thus far given.

13. Define *symmetrical polyhedral angles*. Are symmetrical trihedral angles equal?

14. State theorems on equal and symmetrical trihedral angles.

15. Give examples of loci in plane geometry which differ from corresponding loci in solid geometry. See, for instance, § 6.

MISCELLANEOUS EXERCISES ON BOOK I

1. A Christmas tree is made to stand on a cross-shaped base. What must be the relation of the tree to each piece of the cross in order that it may be perpendicular to the floor?

2. If A, B, and C do not lie in the same line, and if their projections on a plane M do lie in a straight line, what is the relation of the planes M and ABC?

3. Is it possible to project a circle upon a plane so that the projection shall be a straight line-segment? If so, how must the circle and the plane be related?

4. If the projections of a set of points on each of two planes not parallel to each other lie in straight lines, show that the points themselves lie in a straight line.

5. Find the locus of points 3 feet from one of two intersecting planes and 6 feet from the other.

6. It is required that a series of electric lights shall be 7 feet above the floor of a room and 3 feet from the walls. Find the locus of all points at which such lights may be placed.

7. Find the locus of a point in space such that the difference of the squares of its distances from two fixed points, A and B, is constant.

Suggestion. Through either A or B construct a plane $\perp AB$.

8. Given two points A and B on the same side of a plane M. Determine a point P in M such that $AP + PB$ shall be a minimum.

Suggestion. Pass a plane through A and B perpendicular to M, and proceed as in Ex. 10, page 280, Plane Geometry.

9. Show that if the edge of a dihedral angle is cut by two parallel planes, the sections which they make with the faces form equal angles. Is the converse proposition true?

10. Show that if all edges of a trihedral angle are cut by a series of parallel planes, the intersections form a series of similar triangles. Is the converse proposition true?

11. Find the locus of the intersection points of the medians of the triangles obtained in Ex. 10.   Also of the altitudes.

12. How many planes may be made to pass through a given point parallel to a given line?   Discuss the mutual relation of all such planes.

13. Through a point P construct a line meeting each of two lines l_1 and l_2

Suggestion. Let M and N be the planes determined by P and l_1, P and l_2. Consider their intersection. Is this construction always possible?

14. A line l is parallel to a plane M, and lines l_1 and l_2 in M are not parallel to l.   Show that the shortest distance between l and l_1 is equal to the shortest distance between l and l_2.

15. Find the locus of all points equidistant from the planes determined by the faces of a trihedral angle.

16. Find the locus of all points equidistant from the edges of a trihedral angle.

Suggestion. On the edges lay off $PA = PB = PC$. Let O be equidistant from A, B, and C.   Then any point Q in PO is equidistant from the edges.

17. Planes determined by the edges of a trihedral angle and the bisectors of the opposite face angles meet in a line.
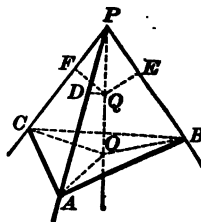
18. If three planes are so related that the segments intercepted on any transversal line are in the same ratio as the segments intercepted on any other transversal then the planes are parallel.

Suggestion. Let M, N, Q be the three planes. From A any point in M draw three lines, not in the same plane, meeting N in A', B', C' and Q in A'', B'', C''. Use the hypothesis to show that $A'B' \parallel A''B''$ and $C'B' \parallel C''B''$.   Hence $Q \parallel N$.   Similarly show that $M \parallel N$.
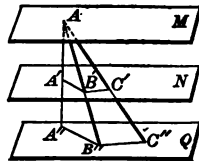
Thales of Miletus (640–542 B.C.) was one of the Seven Wise Men of Greece. He learned astronomy and geometry in Egypt and was the first to introduce the scientific study of geometry in Greece. He measured the height of the Pyramids in Egypt by similar triangles and found a method of computing the distance of a ship at sea. He predicted the solar eclipse of 585 B.C.

BOOK II

REGULAR POLYHEDRONS

153. Polyhedron. A polyhedron is a geometric solid whose boundary consists of plane polygons.

154. Convex Polyhedron. A polyhedron is *convex* if every section of it made by a plane is a convex polygon.
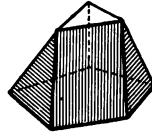
155. Faces, Vertices, Edges. The polygons which bound the polyhedron are its *faces;* the sides of these faces are the *edges* and their vertices are the *vertices* of the polyhedron.

156. Surface of Polyhedron. The faces, edges, and vertices taken together form the surface of the polyhedron.

157. Polyhedrons Classified According to the Number of Faces. A polyhedron of four faces is a *tetrahedron,* one of six faces is a *hexahedron,* one of eight faces an *octahedron,* one of twelve faces a *dodecahedron,* and one of twenty faces an *icosahedron.*

These names are all derived from the Greek and refer to the number of faces. Compare other words derived from the same Greek roots such as hexagon, octagon, dodecagon, etc.

SIGHT WORK

1. What is the smallest number of polygons which may be used to inclose a portion of space ?

Suggestion. Remembering that a triangle is regarded as a polygon, consider a triangular pyramid.

2. How many edges and how many vertices has a tetrahedron ?

3. How many faces, edges, and vertices has a hexahedron ?

Suggestion. Consider a cube.

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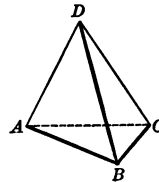
MODELS OF THE REGULAR POLYHEDRONS

158. Regular Polyhedrons. A polyhedron whose polyhedral angles are equal and whose faces are equal regular polygons is a *regular polyhedron.* The following are regular polyhedrons:
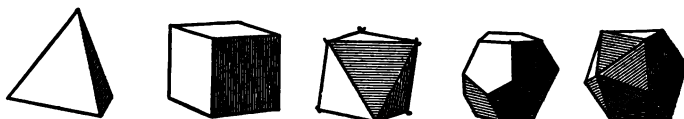
TETRAHEDRON . HEXAHEDRON   OCTAHEDRON   DODECAHEDRON ICOSAHEDRON

159. Models of the Regular Polyhedrons. Models of the regular polyhedrons may be made with cardboard by cutting out patterns, as shown in the figures below, folding along the dotted lines, and fastening the edges together by means of gummed paper strips corresponding to the dotted margins.
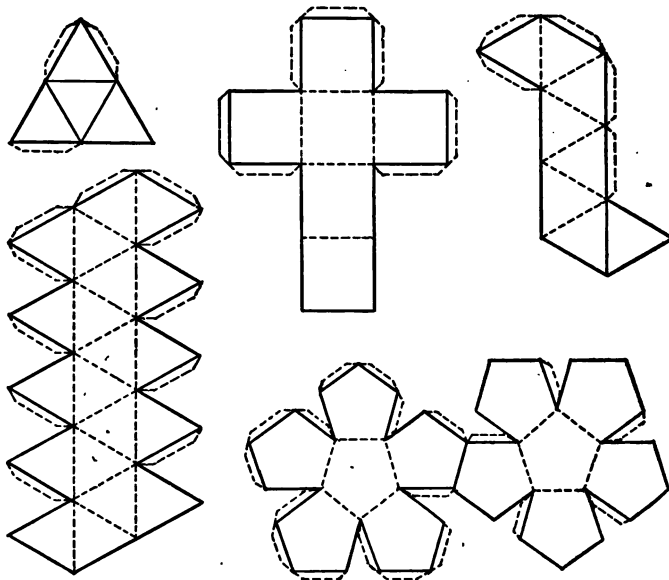
SIGHT WORK

1. What is the smallest possible number of face angles in a polyhedral angle ?

2. How many faces meet in a vertex of a regular tetrahedron ? What is the sum of the face angles forming one of its polyhedral angles ?
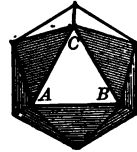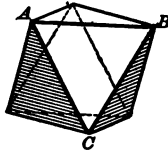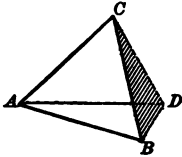
3. How many faces meet in a vertex of a regular octahedron ? in a regular icosahedron ?

4. What is the sum of the face angles forming a polyhedral angle in a regular octahedron ? in a regular icosahedron ?

5. Why may not six or more equilateral triangles be used to form a polyhedral angle ?

6. What is the sum of the face angles forming a polyhedral angle of a cube (hexahedron) ?

7. Why may not four or more squares be used to form a polyhedral angle ?

8. How many faces meet in a vertex of a regular dodecahedron ? What is the sum of the face angles forming one of its polyhedral angles ?
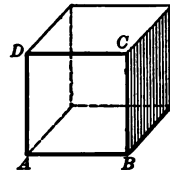
9. Why may not more than three regular pentagons be used to form a polyhedral angle ?

10. May regular hexagons be used to form a polyhedral angle ? Why ?

11. May any regular polygons of more than six sides be used to form polyhedral angles ? Why ?

12. Can we show that not more than three different regular polyhedrons may be formed having triangles as their faces ?
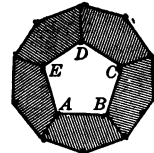
13. How many regular polyhedrons may be formed having squares as their faces ? How many having pentagons as their faces ?

THE NUMBER OF REGULAR POLYHEDRONS

160. THEOREM I. *There are exactly five regular polyhedrons.*

Proof: On page 50 it was shown how models of five different regular polyhedrons may be made.

Hence there are at least five such polyhedrons.

That there cannot be more than five regular polyhedrons follows from the two propositions:

(a) *Every polyhedral angle has at least three face angles.*

(b) *The sum of the face angles of a polyhedral angle is less than 360°.*

From these propositions it follows that each of the polyhedral angles of a regular polyhedron may be formed by three, four, or five (but not six) equilateral triangles ; by three (but not four) squares ; or by three (but not four) regular pentagons. Regular polygons having more than five sides cannot form a polyhedral angle.

The details of the proof are left to the student. The sight work on page 51 will furnish suggestions.

161. Construction of Regular Polyhedrons. The regular polyhedrons may be constructed by use of § 130. The construction for the tetrahedron and octahedron are given below.

(1) **The regular tetrahedron.** At the center E of an equilateral triangle ABC erect a perpendicular to the plane of the triangle. On this take a point D so that $AD = AC$.

Then the four triangles, ABC, ACD, ABD, BCD, are *regular* and equal, and the four trihedral angles are equal. Why?
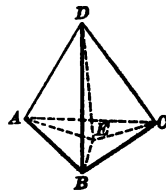
(2) **The regular octahedron.** Through the center O of a square $ABCD$ draw a perpendicular to the plane of the square and on it take points E and F such that $AF = AE = AB$. Join E and F to each of the four vertices, A, B, C, D.

Then the eight faces are equal regular triangles, and the six polyhedral angles are equal. Why?
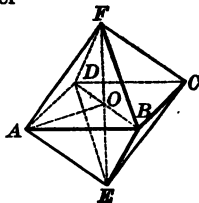
BOOK III

PRISMS AND CYLINDERS

162. Closed Prismatic Surface. Given a convex polygon and a straight line not in its plane. If the line moves so as to remain parallel to itself while it touches the boundary of the polygon and is made to traverse it completely, the line is said to generate *a closed prismatic surface.*

163. Generator. Element. The moving line is the *generator* of the surface, and the generator in any one of its positions is an *element* of the surface.
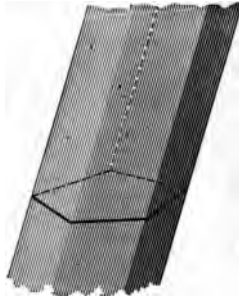
164. Prism. A polyhedron bounded by a prismatic surface and two parallel plane sections cutting all its elements is called a *prism.*

165. Bases. Lateral Surface. Edges. The two parallel cross-sections which bound a prism are its *bases* and the other faces form its *lateral surface.* The *edges* are the lines in which its lateral faces meet.
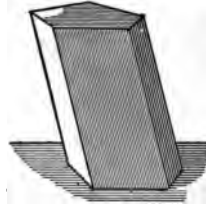
166. Right Section. Right Prism. A *right section* of a prism cuts all its edges at right angles. A *right prism* is one whose bases are right sections.

167. COROLLARY 1. *The lateral faces of a prism are parallelograms.*

168. COROLLARY 2. *The lateral edges of a prism are equal and parallel.*

169. COROLLARY 3. *The lateral faces of a right prism are rectangles.*

53
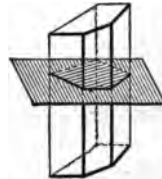
CLASSIFICATION OF PRISMS

170. Kinds of Prisms. Prisms are classified according to the form of their right sections, as *triangular, quadrangular, pentagonal, hexagonal,* etc. A *regular prism* is one whose right section is a regular polygon.

171. Altitude. Area. The *altitude* of a prism is the perpendicular distance between the planes of its bases. The altitude of a right prism is equal to its edge.

The *lateral area* of a prism is the sum of the areas of its lateral faces.

The *total area* is the sum of its lateral area and the areas of its bases.
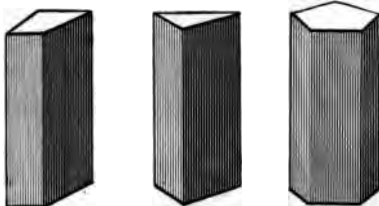
172. Truncated Prism. A polyhedron which is a part of a prism cut off by a plane meeting all the lateral edges, but not parallel to the base, is called a *truncated prism.*

Two polyhedrons are said to be *added* when they are placed so that a face of one coincides wholly or in part with a face of the other, but otherwise each lies outside the other.
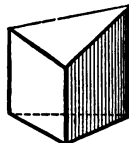
173. Parallelopiped. A *parallelopiped* is a prism whose bases, as well as lateral faces, are parallelograms.

A *rectangular parallelopiped* has its bases and all its faces rectangles.

A *cube* is a parallelopiped whose bases and faces are all squares.
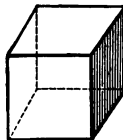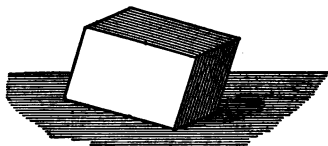
174. THEOREM I. *The cross-sections of a prism made by parallel planes are equal polygons.*
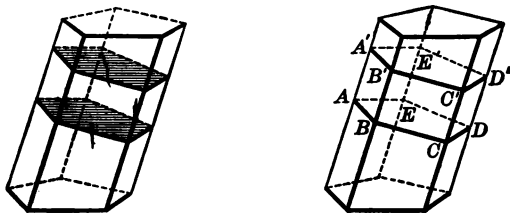
Given a prism cut by two parallel planes forming the polygons *ABCDE* and *A'B'C'D'E'*.

To prove that $ABCDE = A'B'C'D'E'$.

Outline of Proof: (1) Show that $AB = A'B'$, $BC = B'C'$, etc., by proving that $ABB'A'$, $BCC'B'$, etc., are $\boxed{s}$.

(2) Show that $\angle ABC = \angle A'B'C'$, $\angle BCD = \angle B'C'D'$, etc.

(3) Hence show that $ABCDE$ and $A'B'C'D'E'$ can be made to coincide.

175. COROLLARY. *Every cross-section of a prism parallel to the base is equal to the base.*

SIGHT WORK

1. Can a parallelopiped be a *right* prism without being a *rectangular* parallelopiped? Illustrate.

2. Show that in a rectangular parallelopiped each edge is perpendicular to the other edges which meet it.

3. Show that any section of a prism made by a plane parallel to an edge is a parallelogram.

Suggestion. Use in order §§ 89, 84, and 92.

4. Is it possible to construct a prism such that there is no plane perpendicular to its edges and cutting all of them unless some of them are extended?

THE LATERAL AREA OF A PRISM

176. THEOREM II. *The lateral area of a prism is equal to the product of a lateral edge and the perimeter of a right section.*
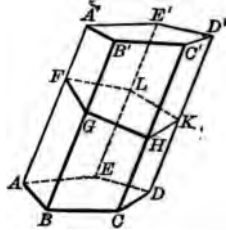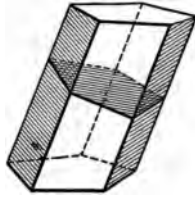
Suggestion. Show that the lateral edges are mutually equal and that the area of each face is the product of a lateral edge and one side of the right-section polygon.

Complete the proof.

Note. The form of statement in this theorem is the usual abbreviation for the more *precise* form :

The lateral area of a prism is equal to the product of the *numerical measures* of a lateral edge and the perimeter of a right section.

Similar abbreviations are used throughout this text.

177. COROLLARY. *The lateral area of a right prism is equal to the product of its altitude and the perimeter of its base.*

SIGHT WORK

1. If the lateral edge of a prism is 8 inches and the perimeter of its right section 30 inches, what is the lateral area ?

2. Of all prisms having an altitude 6 inches and the perimeter of a right section 24 inches, which one has the smallest lateral area ?

Suggestion. Which one has the shortest lateral edge ?

3. Why is not the lateral area of an oblique prism determined by the perimeter of the base and a lateral edge ?

178. THEOREM III. *Two prisms are equal if three faces having a common vertex in the one are equal respectively to three faces having a common vertex in the other, and similarly placed.*
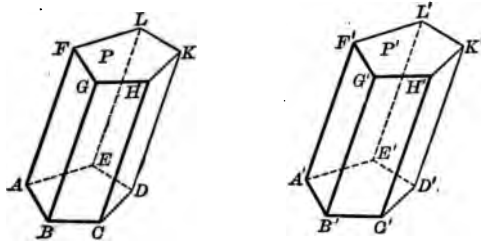
Given the three faces meeting at *B* in prism *P* equal respectively to the three faces meeting at *B'* in prism *P'*, and similarly placed.

To prove that P can be made to coincide with P'.

Outline of Proof: Trihedral ∠ *B* and *B'* are equal.   Why ?

Now apply the two prisms, making *B* coincide with *B'*, and then show in detail that:

(1) The lower bases coincide.

(2) The lateral faces at *B* and *B'* coincide.

(3) The upper bases coincide.

(4) All the lateral faces coincide.

179. COROLLARY 1. *Two truncated prisms are equal under the conditions of* § 178.
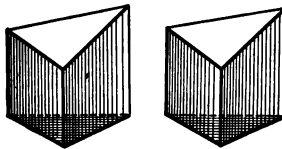
180. COROLLARY 2. *Two right prisms are equal if they have equal bases and equal altitudes.*

OPPOSITE FACES OF A PARALLELOPIPED

181. THEOREM IV. *Opposite faces of a parallelo-piped are equal and parallel.*
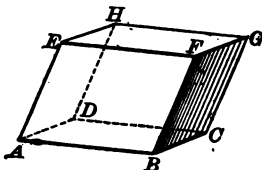
Suggestions. Consider the opposite faces *ABFE* and *DCGH.*

(1) Show that the sides of the angles *ABF* and *DCG* are parallel, and hence that the planes determined by them are parallel.

(2) Show that these faces are equal.

In like manner argue about any other pair of opposite faces.

182. COROLLARY. *Any section of a parallelopiped made by a plane cutting four parallel edges is a parallelogram.*

Suggestion. In the figure show that *RQ ‖ ST* and *QT ‖ RS.*
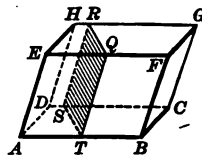
SIGHT WORK

1. A section made by a plane passed through two diagonally opposite edges of a parallelopiped is a parallelogram, for example, the section through *DH* and *BF* in the figure of § 181.

2. Show that any two of the four diagonals of a parallelopiped bisect each other.

Suggestion. Make use of example 1.

3. If the parallelopiped in § 182 is rectangular, is the section necessarily a rectangle ? May it be so ?

4. Show that the section of a cube made by a plane through two diagonally opposite edges is a rectangle. Why can it not be a square ? Can you construct a parallelopiped in which such a section would be a square ?

PLANE THROUGH OPPOSITE EDGES OF A PARALLELOPIPED

183. THEOREM V. *A plane through diagonally opposite edges of a parallelopiped divides it into two triangular prisms having equal right sections and equal lateral edges.*
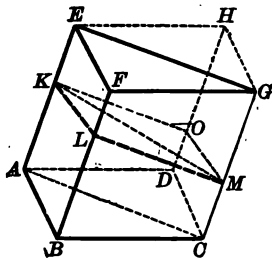
Given the parallelopiped *ABCD-F* with opposite edges *AE* and *CG*.

To prove that the plane through AE and CG divides it into two triangular prisms having equal lateral edges and equal right sections.

Proof: By § 182 the right section *KLMO* of the given prism is a parallelogram. Hence *KM* divides it into two equal triangles. Hence the right sections of the prisms *ABC–F* and *ACD–H* are equal.

The lateral edges *BF* and *DH* are equal by § 168.

SIGHT WORK

1. If a right section of a prismatic space is a parallelogram, what kind of prism will be cut out by two right sections?

2. If a right section of a prismatic space is a rectangle, what kind of prism will be cut out by two right sections?

3. If a right section of a prismatic space is a regular hexagon, what kind of prism will be cut out by two right sections?

4. What kind of prism will be cut out by two right sections of any prismatic space?

1. The face angles of one trihedral angle of a parallelopiped are 65°, 70°, and 75° respectively. What are the face angles of the other trihedral angles?

2. Show that two prisms cut from the same prismatic space and having equal lateral edges have equal lateral areas.

3. Show that the diagonals of a rectangular parallelopiped are all equal to each other.

4. Show that the square on the diagonal of a rectangular parallelopiped is equal to the sum of the squares on the sides meeting at a vertex from which it is drawn.
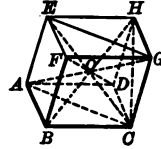
E.g. in the figure $\overline{AG}^2 = \overline{AC}^2 + \overline{CG}^2 = \overline{AB}^2 + \overline{BC}^2 + \overline{CG}^2$.

5. Find the ratio of the diagonal to one edge of a cube.

6. Find the edge of a cube whose diagonal is 14 inches. Find the diagonal of a cube whose edge is 16 inches.

7. Find a diagonal of a rectangular parallelopiped whose edges are 6, 8, and 10 inches respectively.

8. Are the diagonals of a cube perpendicular to each other?

9. In a rectangular parallelopiped the diagonal of the base is 12 in. and the altitude is 5 in. Find the diagonal of the parallelopiped.

10. If two equal right prisms whose bases are equilateral triangles are placed together so as to form one prism whose base is a parallelogram, compare the lateral area of the prism so formed with the sum of the lateral areas of the original prisms.

11. A right prism whose bases are regular hexagons is divided into six prisms whose bases are equilateral triangles. Compare the lateral area of the original prism with the sum of the lateral areas of the resulting prisms.

VOLUME OF A RECTANGULAR PARALLELOPIPED

184. Thus far certain properties of prisms have been studied, but no attempt has been made to *measure the space occupied* by such a solid. For this purpose we consider first a rectangular parallelopiped.

185. Numerical Measure. In case each edge of a rectangular parallelopiped is commensurable with a unit segment, the number of times which a unit cube is contained in it is the *numerical measure* or the *volume* of the parallelopiped.

186. The Commensurable Case. In the commensurable case just described, the volume is easily computed.
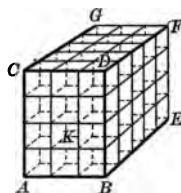
E.g. if in the figure one edge *AC* is 4 units, and an adjoining edge *AB* is 3 units, then a cube as *AK*, whose edge is one unit, may be laid off 4 times along *AC* and a tier of $3 \cdot 4 = 12$ such cubes will adjoin the face *AD*. Since the edge *BE* is 5 units long, 5 such tiers will exactly fill the space within the solid. That is, $5 \cdot 3 \cdot 4 = 60$ is the number of cubic units in the solid.

Hence in this case

$$Volume = Length \times Width \times Height.$$

Again, if the given dimensions are 3.4, 2.6, 4.5 decimeters respectively, then unit cubes, with edge one *decimeter*, cannot be made to fill exactly the space inclosed by the figure, but cubes with edge each one *centimeter* will do so, giving 34, 26, and 45 respectively along the three edges of the figure.

Therefore, the volume is

$34 \cdot 26 \cdot 45 = 39{,}780$ cubic centimeters, or 39.78 cubic decimeters.

Hence in this case also

$$Volume = Length \times Width \times Height.$$

187. Formula for Volume of a Rectangular Solid. We therefore conclude that the volume of a rectangular parallelopiped with commensurable dimensions is given by the formula

$$\textbf{Volume} = \textbf{Length} \times \textbf{Width} \times \textbf{Height.}$$

VOLUME OF A RECTANGULAR SOLID

188. The Incommensurable Case. If a rectangular parallelo-piped is such that any two of its dimensions are incommensurable with each other, then there is no unit cube, however small, in terms of which the volume can be exactly expressed.

But by choosing a unit cube sufficiently small we may determine the volume of a rectangular solid which differs from the given one by as little as we please.

E.g., if the length is 5 inches and the width is $\sqrt{3}$ inches, then the base cannot be exactly covered with equal cubes, however small.

But since $\sqrt{3} = 1.732 \cdots$, if we take as a unit of volume a cube one one-hundredth of an inch on a side, then the base of a rectangular solid whose length is 5 inches and whose width is 1.73 inches can be *exactly covered* with a layer of these cubes and the number of cubes in such a layer is

$$500 \times 173 = 86,500.$$

If the height is 2 inches, then 200 layers will just reach the top, making

$$500 \times 173 \times 200 = 17,300,000 \text{ cubes.}$$

And since 1,000,000 such small cubes make a cubic inch, the volume of this solid is 17.3 cubic inches.

The portion of the given solid not thus filled is 5 inches long, 2 inches high and less than .003 of an inch in thickness.

Hence, its volume is less than $5 \times 2 \times .003 = .03$ of a cubic inch.

By expressing $\sqrt{3}$ to further places of decimals and then using smaller and smaller units of volume, successive rectangular solids may be found which differ less and less from the given solid.

The foregoing considerations constitute an informal proof of the following theorem:

189. THEOREM VI. *The volume of a rectangular parallelopiped is equal to the product of its length, width, and height.*
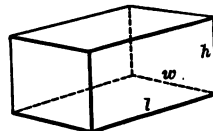
In symbols, $V = l \cdot w \cdot h$.

The above argument shows that this theorem holds for all rectangular parallelopipeds used in the process of approximation, and hence it applies to all practical measurements of the volumes of such solids.

190. COROLLARY 1. *The volume of a rectangular parallelopiped is equal to the product of the numerical measures of its base and altitude.*

191. COROLLARY 2. *If two rectangular parallelopipeds have two dimensions respectively equal to each other, their volumes are in the same ratio as their third dimensions; and if they have one dimension the same in each, their volumes are in the same ratio as the products of their other two dimensions.*

For if V and V' are the volumes, and a, b, c, and a', b', c' the dimensions, then $\dfrac{V}{V'} = \dfrac{a \cdot b \cdot c}{a' \cdot b' \cdot c'} = \dfrac{a}{a'}$ if $b = b'$ and $c = c'$; or $\dfrac{V}{V'} = \dfrac{a \cdot b}{a' \cdot b'}$ if $c = c'$.

192. Volumes of Prisms in General. From the formula for volumes of rectangular solids, $V = l \cdot w \cdot h$, we deduce the volumes of prisms in general by means of the principle:

Two polyhedrons have the same volume if they can be made to coincide, or if they can be divided into parts which can be made to coincide in pairs.

The sign $=$ between two polyhedrons means that they are equal in all respects; that is, can be made to coincide. The word *equivalent* is used to mean *equal in volume*.

EXERCISES

1. The dimensions in inches of a rectangular parallelopiped are 5, 10, and $2\sqrt{2}$. Taking the approximate value of $\sqrt{2}$ as 1.4142, find the approximate volume. Show that this differs from the exact volume by less than .001 of a cubic inch, it being given that $\sqrt{2}$ lies between 1.41421 and 1.41422.

2. If the dimensions in inches of a rectangular parallelopiped are 5, $4\sqrt{2}$, and $5\sqrt{3}$, find how near an approximation to the volume is possible by taking $\sqrt{2}$ between 1.414 and 1.4143 and $\sqrt{3}$ between 1.732 and 1.7321.

3. If the dimensions in inches of a rectangular parallelopiped are $3\sqrt{2}$, $3\sqrt{3}$, $2\sqrt{5}$, find the values to be used for $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, to obtain the volume within .001 of a cubic inch.

VOLUME OF AN OBLIQUE PRISM

193. THEOREM VII. *The volume of an oblique prism is equal to that of a right prism having for its base a right section of the oblique prism and for its altitude a lateral edge of the oblique prism.*
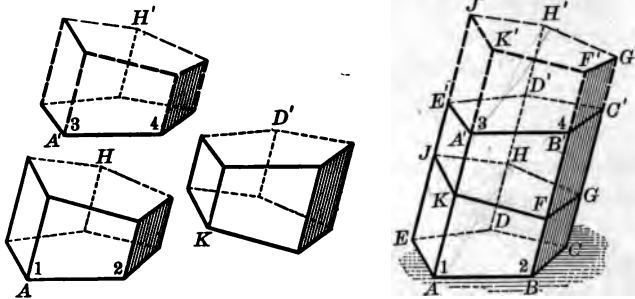
Given the oblique prism AD' with $FGHJK$ a right section and $F'G'H'J'K'$ a right section of the prism extended so that $AA' = KK'$.

To prove that the oblique prism AD' has the same volume as the right prism KH'.

Proof: In the truncated prisms AH and $A'H'$,
base $ABCDE = $ base $A'B'C'D'E'$.   § 175

In faces $ABFK$ and $A'B'F'K'$, we have
$\qquad AB = A'B', \angle 1 = \angle 3, \angle 2 = \angle 4,$ **Why?**
and $\qquad AK = A'K'$ and $BF = B'F'.$ **Why?**
$\therefore ABFK$ and $A'B'F'K'$ will coincide and are equal.
Likewise, $\qquad\qquad AKJE = A'K'J'E'.$
$\qquad\qquad\qquad \therefore AH = A'H'.$ § 179
But the given prism $\quad AD' = AH + KD',$
and the right prism $\qquad KH' = A'H' + KD'.$
$\qquad\qquad\qquad \therefore AD' = KH'.$ Q. E. D.

194. COROLLARY. *Two prisms having equal lateral edges and equal right sections are equal in volume.*

VOLUME OF ANY PARALLELOPIPED

195. THEOREM VIII. *The volume of any parallel-opiped is equal to the product of its base and altitude.*
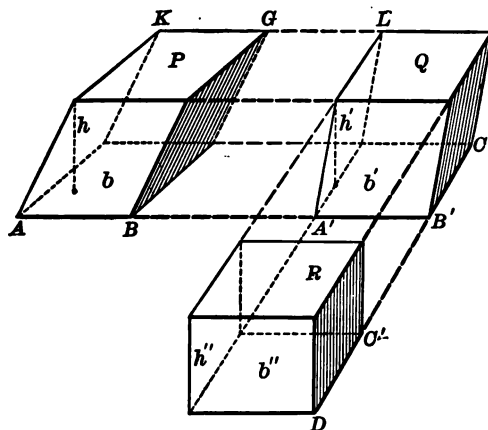
Given any parallelopiped *P* **with area of base** *b* **and altitude** *h.*

To prove that its volume is equal to $b \times h.$

Proof: Considering face *AK* as the base of prism *P*, produce the four edges parallel to *AB*, and lay off $A'B' = AB.$

Through *A'* and *B'* erect planes $\perp A'B'$, thus cutting off the right prism *Q* with *A'L* as one base.

Now considering *CL* as the base of prism *Q*, produce the four edges ∥ *CB'* and lay off $C'D = CB'.$

At *C'* and *D* erect planes $\perp C'D$, cutting off the right prism *R*, which is a rectangular parallelopiped with base *b''.*

Now prove in detail: (1) $h = h' = h''$, (2) $b = b' = b''$, (3) Vol. $P =$ Vol. $Q =$ Vol. *R*, § 193 (4) Vol. $R = b''h''$, (5) Vol. $P = bh.$

196. COROLLARY. *Two parallelopipeds are equal in volume if they have equal altitudes and bases of equal areas.*

197. THEOREM IX. *The volume of a triangular prism is equal to the product of its base and altitude.*
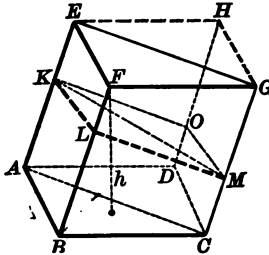
Given the triangular prism whose base is *ABC*.

To prove that the volume is equal to the area of $\triangle ABC$ multiplied by the altitude h.

Proof: Complete the ▱ *ABCD* and *EFGH* and draw *DH*. Now show that *CDHG* and *ADHE* are ▱, and hence that *BH* is a parallelopiped.   Use § 43 (2)

Let *KLMO* be a right section of *BH*, and let *KM* be the line in which the plane *ACGE* cuts the plane *KLMO*.

Then (1) prism *ABC–F* = ½ prism *BH*,   §§ 183, 194

(2) But prism *BH* = h × area of ▱ *ABCD*,   § 196

(3) Hence, prism *ABC–F* = h × ½ area of ▱ *ABCD*, that is, prism *ABC–F* = h × area of $\triangle ABC$.

Therefore the volume of prism *ABC–F* is equal to the area of its base times its altitude.   Q. E. D.

SIGHT WORK

1. Find the volume of a prism whose altitude is 8 in. and whose base is a right triangle with legs 5 in. and 6 in. respectively.

2. Find the volume of a right prism whose altitude is 6 in. and whose base is a rectangle with sides 3 in. and 5 in.

VOLUME OF ANY PRISM

198. THEOREM X. *The volume of any prism is equal to the product of its base and altitude.*
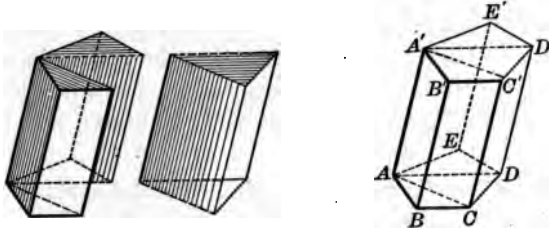
Outline of Proof: Any prism can be divided into triangular prisms by planes passing through one edge and each of the other non-adjacent edges.

The altitudes of the triangular prisms are the same as that of the given prism, and the sum of their bases is equal to the base of the given prism.

Now use § 197 and complete the proof.

199. COROLLARY 1. *The product of the base and altitude of any prism is equal to the product of a lateral edge and the area of a right section.*

For each equals the volume of the prism. See §§ 198, 193.

200. COROLLARY 2. *If two prisms have equivalent bases, their volumes are in the same ratio as their altitudes; and if they have equal altitudes, their volumes are in the same ratio as the areas of their bases.*

SIGHT WORK

1. Find the volume of a prism whose altitude is 8 in. and the area of whose base is 42 sq. in.

2. The volume of a prism is 264 cu. in. and its altitude is 8 in. Find the area of its base.

EXERCISES

1. The theorem that the volume of any prism is equal to the product of its base and altitude is of great importance. What theorems of Book III have been used directly or indirectly in proving it?

2. What dimensions of a prism must be known in order to determine its lateral area by means of the preceding theorems of Book III? What dimensions must be known to determine its volume?

3. The edge of a cube is e. Find the total surface and the volume in terms of e.

4. Find the edge of a cube if its total area is equal numerically to its volume, an inch being used as the unit.

5. Find the volume of a regular right triangular prism whose edges are all equal to 6 inches.

6. Find the volume of a regular right hexagonal prism whose edges are all equal to 10 inches.

7. A side of the base of a regular right hexagonal prism is 3 inches. Find its altitude if its volume is $54\sqrt{3}$ cubic inches. What is the total area of this prism?

8. The volume of a triangular prism is equal to the area of one lateral face multiplied by half the perpendicular distance of this face from the remaining edge. Prove.

9. The volume of a regular right prism is equal to the lateral area multiplied by half the apothem of the base. Prove.

10. Prove that the sum of the squares of the four diagonals of a rectangular parallelopiped is equal to the sum of the squares of the twelve edges of the parallelopiped.

11. A prismatic space is cut by two pairs of parallel planes. Prove that the volumes of the two prisms so formed are equal if the two pairs of planes are the same distance apart.

CYLINDERS

201. Curved Surface. A surface no part of which is plane is called a *curved surface*.

E.g. the surface of an eggshell or of a stovepipe is a curved surface.

202. A Closed Plane Curve. A curve which can be traced continuously by a point moving in a plane so as to return to its original position without crossing its path is *a closed plane curve*.

A closed plane curve is *convex* if it can be cut by a straight line in not more than two points.

203. A Cylindrical Surface. If a straight line moves so as to remain parallel to itself, while it always touches a closed convex plane curve and is made to traverse it completely, the line is said to generate a *closed convex cylindrical surface*. The moving line is the *generator*, and the generator in any one of its positions is an *element* of the surface.
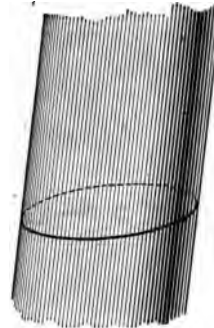
204. Cylinder. A solid bounded by a cylindrical surface and two parallel plane sections cutting all its elements is called a *cylinder*.

205. Bases. Lateral Surface. The two parallel cross-sections which bound a cylinder are its *bases*, and the curved surface is its *lateral surface*.

206. Element. Altitude. That part of the generator of a cylindrical surface which is included between the bases of a cylinder is called an *element* of the cylinder. The *altitude* of a cylinder is the perpendicular distance between its bases.

CIRCULAR CYLINDERS

207. Right Section. A *right section of a cylinder* is made by a plane cutting each of its elements at right angles.

208. Circular Cylinders. A *circular cylinder* is one whose right section is a circle.

The radius of a circular cylinder is the radius of its right section.

A cylinder whose elements are at right angles to its bases is called a *right cylinder.* Otherwise it is an *oblique cylinder.* A right cylinder whose bases are circles is a *right circular cylinder.*
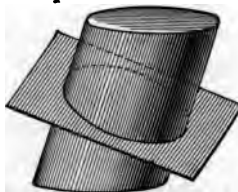
209. Axis of a Cylinder. The line passing through the centers of two right sections of a circular cylinder is the *axis* of the cylinder.

A right circular cylinder may be generated by revolving a rectangle about one of its sides as an axis. The side opposite the axis generates the lateral surface, and the sides adjacent to the axis generate the bases.

SIGHT WORK

1. What is the locus of all points in space which are at a perpendicular distance of 6 in. from a straight line 10 in. long?

2. Show that any two elements of a cylinder determine a plane section of the cylinder which cuts the two bases in parallel lines.

3. If every plane determined by two elements of a cylinder is perpendicular to its bases, what kind of cylinder is it?

4. If one right section of a cylinder is a circle, what would appear to be true of all its right sections? For proof see § 213.

5. If a circular cylinder is oblique, are its bases circles?

6. If an oblique cylinder has a circular base, is its right section a circle?

LONGITUDINAL SECTIONS OF A CYLINDER

210. THEOREM XI. *If a plane contains an element of a cylinder and meets it in any other point, then it contains another element also, and the section is a parallelogram.*
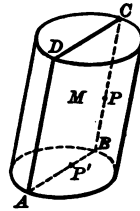
Given the cylinder AC and a plane M containing the element AD and another point as P or P'.

To prove that it contains another element and that the section containing these two elements is a parallelogram.

Proof: (1) *When the point P is in the lateral surface.*

Let BC be the element through P.

Then $BC \parallel AD$ and they determine a plane M.　　§§ 203, 73

Also $AB \parallel DC$ and $ABCD$ is a parallelogram.　　§ 92

(2) *When the point P' is in one of the bases.*

Draw AB through P'. Then by (1) the plane M contains the element BC and the section is a parallelogram.　　Q. E. D.
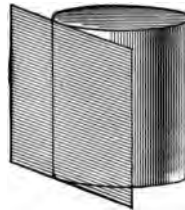
211. Plane Tangent to a Cylinder. If a plane contains an element of a cylinder and no other point in it, the plane is said to be *tangent to the cylinder*, and the element is called the *line of contact*.

It may be noted that a cross-section of a cylinder and its tangent plane consists of a plane curve and a line tangent to it.

212. THEOREM XII.   *The bases of a cylinder are equal plane figures.*
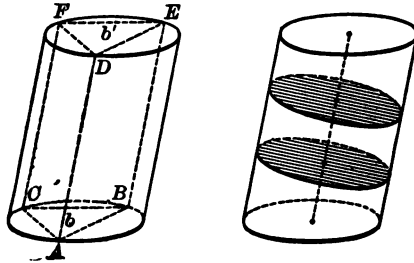
Given a cylinder with the bases *b* and *b'*.

To prove that b = b'.

Proof: Take any three points *A, B, C* in the rim of the base *b* and draw elements through these points, meeting the base *b'* in *D, E, F*.

Show that △ *ABC* = △ *DEF*.   Use §§ 210, 27

Now, while the elements *AD* and *BE* remain fixed, conceive *CF* to generate the surface of the cylinder.

Evidently △ *ABC* = △ *DEF* in every position of *CF*.

Hence, if base *b'* is applied to base *b* with these triangles coinciding in *one* position, they will coincide in every position corresponding to the moving generator.

That is, *b'* coincides with *b*.   Q. E. D.

The proposition of § 212 may also be stated in the form of the following corollary :

213. COROLLARY 1.   *Parallel plane sections of a cylinder are equal, if they cut all the elements.*

214. COROLLARY 2.   *The axis of a circular cylinder passes through the center of all its right sections.*

MEASURING THE SURFACE AND VOLUME OF A CYLINDER

215. Area of a Curved Surface. Thus far in geometry the word *area* has been used in connection with plane figures only. In some cases the computation of an area has been made by an approximation process only, as in the case of some rectangles and of the circle. Indeed, in these cases approximate measurement only is possible, since no square unit exists in terms of which such areas can be *exactly* measured.

In the case of *any curved surface* it is evident that approximate measurement is the *only kind possible* in terms of a plane area unit, since no such unit, however small, can be made to coincide with a part of such a surface.

216. Volume of a Solid Having a Curved Surface. Since the unit of measure for solids is the cube, it is evident that a solid having a curved surface cannot contain such a unit an integral number of times. Hence approximate measurement only of the volumes of such solids is possible.

To measure approximately the area and volume of a cylinder use is made of inscribed and circumscribed prisms.

217. Inscribed Prisms. A prism is said to be *inscribed in a cylinder* if its lateral edges are elements of the cylinder, and their bases lie in the same planes.
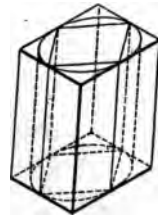
218. Circumscribed Prisms. A prism is said to be *circumscribed about a cylinder* if its lateral faces are all tangent to the cylinder, and their bases lie in the same planes.

It is evident that by increasing the number of lateral faces of the inscribed or circumscribed prisms the surface of the latter may be made to lie as near the surface of the cylinder as we please. The lateral edges of the prisms will remain equal to an element of the cylinder, while the right sections of the prisms can be made to differ as little as we please from the right section of the cylinder.

LATERAL AREA OF A CYLINDER

219. Fundamental Assumption on the Area and Volume of a Cylinder. We now assume that

A cylinder has a definite lateral area and a definite volume which may be approximated as nearly as we please by taking the lateral areas and the volumes of successive inscribed or circumscribed prisms.
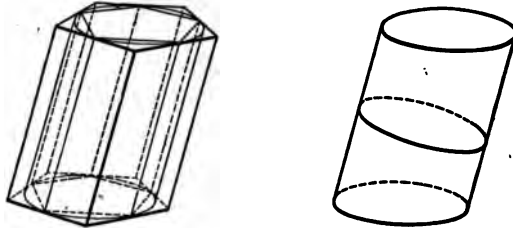
220. Lateral Area of a Cylinder. Since *the lateral area of a prism is equal to the product of a lateral edge and the perimeter of a right section* (§ 176), it follows that this theorem holds for every inscribed or circumscribed prism used in approximating the lateral area of a cylinder.

The foregoing considerations constitute an informal proof of the following theorem:

221. THEOREM XIII. *The lateral area of a cylinder is equal to the product of an element of the cylinder and the perimeter of a right section.*

222. COROLLARY 1. *If r is the radius of a right section of a circular cylinder, and e is an element, the lateral surface is* $S = 2\,\pi r e$ (§ 176).

223. COROLLARY 2. *If r is the radius and h is the altitude of a right circular cylinder, then the lateral surface is* $S = 2\,\pi r h$ (§ 177).
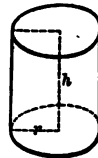
VOLUME OF A CYLINDER

224. Since *the volume of a prism is equal to the product of its altitude and the area of its base,* it follows that this theorem holds for every inscribed or circumscribed prism used in approximating the volume of a cylinder.

All practical computations of the surfaces and the volumes of cylinders are *approximations* based upon the propositions stated in §§ 220, 224.

The foregoing constitutes an informal proof of the following theorem :

225. THEOREM XIV. *The volume of a cylinder is equal to the product of its altitude and the area of its base.*

226. COROLLARY 1. *If r_1 is the radius of the right section of a circular cylinder and e is an element, then the volume is* (§ 199) $V = \pi r_1^2 e.$
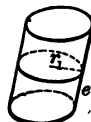
227. COROLLARY 2. *If a cylinder of altitude h has a circular base whose radius is r_2, then the volume is*

$$V = \pi r_2^2 h.$$

228. COROLLARY 3. *If h is the altitude and r the radius of the base of a right circular cylinder, then the volume is* $V = \pi r^2 h.$
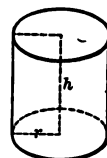
In this case $r = r_1 = r_2$, and $e = h$, and the formulas of corollaries 1 and 2 become identical.

Note. The theorems of §§ 221, 225 were stated (and are true) for any cylinders whatever. However, the theorem of § 225 is available for the computation of the volume of a cylinder only in case the area of the base or of the right section can be computed. And this is possible by *elementary methods* only in case these are circles.

The lateral area of a cylinder can be computed only in case the perimeter of a right section can be found, and this is possible by elementary methods only in case the right section is a circle.

SIGHT WORK

1. If a prism is inscribed in a cylinder, is its lateral area greater or less than that of the cylinder ? Is its volume greater or less than that of the cylinder ?

2. If a prism of say 1000 faces is inscribed in a cylinder, would the theorems of § 221 and § 225 apply directly to this prism ? Would it be easy to detect experimentally the difference between this prism and the cylinder ?

3. How would the right section of the prism and the cylinder of example 2 be related ?

4. Discuss the questions in examples 1, 2, 3 as related to prisms circumscribed about a cylinder.

5. What is the lateral area of a cylinder whose element is 8 inches and the perimeter of whose right section is 24 inches ?

6. What is the volume of a cylinder whose altitude is 10 inches and the area of whose base is 50 square inches ?

7. What is the volume of a cylinder whose edge is 4 inches, and the area of whose right section is 24 square inches ?

EXERCISES

1. Find the volume of a right circular cylinder with radius 5 and altitude 8.

2. Find the volume of a circular cylinder if the radius of a right section is 6 and the length of an element is 10.

3. Find the lateral area of a cylinder if the perimeter of a right section is 39 and the length of an element is 8.

4. Find the lateral area of a right cylinder if the perimeter of the base is 30 and the altitude is 5.

5. Find the total surface area of a right circular cylinder of radius 3 and altitude 5.

6. Find the volume of an oblique cylinder whose base is a circle of radius 4 and whose altitude is 8.

SUMMARY OF BOOK III

1. Define prismatic surface and prismatic space.

2. Define prism, base of prism, lateral surface, right section, right prism, parallelopiped, truncated prism.

3. Give the rule for finding the lateral area of an oblique prism. How may this rule be modified in the case of a right prism?

4. What can be said about opposite faces of a parallelopiped as to shape and size?

5. Give two rules for finding the volume of an oblique prism. See §§ 193, 198.

6. In proving the theorem on the volume of a prism, we considered triangular prisms, general parallelopipeds, rectangular parallelopipeds, and general prisms. In what order were these taken? Why?

7. Show how you would find the approximate volume of a rectangular parallelopiped whose dimensions are $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$. Find this volume accurately to two places of decimals. Discuss the problem of finding this volume correct to four places of decimals. See exercises on page 63.

8. Define closed plane curve, convex plane curve, cylindrical surface, cylindrical space, cylinder, right cylinder, circular cylinder, tangent plane to a cylinder.

9. Give the rule for finding the lateral surface of any cylinder, of a right circular cylinder.

10. Give two rules for finding the volume of an oblique cylinder. Show that these both hold true for a right circular cylinder.

11. Describe what is meant by the statement "the area and the volume of a cylinder may be approximated by taking the area and the volume of inscribed or circumscribed prisms."

MISCELLANEOUS EXERCISES ON BOOK III

1. If the lateral surface of a cylinder and the length of an element are known, can the perimeter of a right section be found? If the lateral area is 400 π, and an element 15, find the perimeter of a right section.

2. The diameter of a right circular cylinder is 8, and the diagonal of the largest rectangle made by a vertical section is 16. Find its altitude.

3. The volume of a right circular cylinder is 128 π cubic inches and its altitude is equal to its diameter. Find the altitude and the diameter.

4. A rectangle whose sides are a and b is turned about the side a as an axis and then about the side b. Find the ratio of the volumes of the two cylinders thus developed. Also find the ratio of the total surfaces of these cylinders.

5. Find the diameter of a right circular cylinder if its lateral area is equal numerically to its volume. Does the result depend upon the altitude of the cylinder?

6. If the altitude of a right circular cylinder is equal to its diameter, find the ratio of the numerical values of its total area and its volume. Does this depend on the radius?

7. A regular hexagonal prism is inscribed in a right circular cylinder whose altitude is equal to the diameter. Find the difference between the volumes of the cylinder and the prism, if a side of the hexagon is 4 inches.

8. A cylindrical tank 8 feet in diameter, partly filled with water, is lying on its side. If the greatest depth of the water is 6 feet, what fraction of the volume of the tank is filled with water?
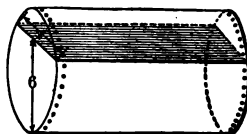
9. In the preceding problem find the fraction of the volume occupied by water if the width of the top of the water along a right cross-section of the tank is 4 feet.

BOOK IV

PYRAMIDS AND CONES

229. Pyramidal Surface. Given a convex polygon and a fixed point P not in its plane. If a line through the fixed point moves so as to touch the boundary of the polygon and is made to traverse it completely, the line is said to generate a convex *pyramidal surface.*

The moving line is the *generator* of the surface, and in any of its positions it is an *element* of the surface. The fixed point is the *vertex* of the pyramidal surface.
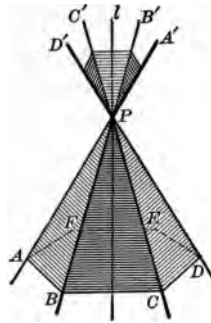
230. Nappes. A pyramidal surface has two parts, one on each side of the vertex, which are called *nappes.*

231. Pyramid. The solid bounded by a pyramidal surface and a plane section cutting all the elements, and not passing through the vertex, is called a *pyramid.*
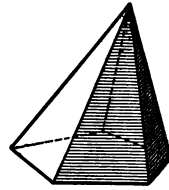
SIGHT WORK

If a plane cuts both nappes of a pyramidal surface, is it possible that it should cut every element of the surface?

Suggestion. If a plane M cuts both nappes, then a plane through the vertex $\parallel M$ meets the surface in two elements which are $\parallel M$ and hence are not cut by M.

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CLASSIFICATION OF PYRAMIDS

232. Faces. Edges. The lateral surface of a pyr-
amid is composed of *triangular faces* having a
common vertex at the vertex of the pyramid, and
having as bases the sides of the polygon forming
the base. The sides common to two such triangles
are the *edges* of the pyramid.

Pyramids are classified, according to the shape of the base, as
triangular, quadrangular, pentagonal, etc.

233. Tetrahedron. A pyramid having a triangular base has
in all four faces, and is called a *tetrahedron.* In this case
every face is a triangle, and any one of them may be taken as
the base.

The **altitude of a pyramid** is the perpendicular distance from
the vertex to the plane of the base.

234. Regular Pyramid. A *regular pyramid* is one whose base
is a regular polygon such that the perpendicular from the ver-
tex upon it meets it at the center.

235. Properties of a Regular Pyramid.

(1) *The edges are equal to each other.*

For they cut off equal distances from the foot of the perpendicular.

(2) *The faces are equal isosceles triangles.*

(3) *The altitudes of the triangular faces are equal to each other.*

236. Slant Height. The altitude of any one of the triangular
faces of a regular pyramid is called its *slant height.*

SIGHT WORK

1. If the slant height of a regular pyramid and the apothem of its base
are given, how may its altitude be computed? If the altitude and
apothem are given, how may the slant height be found?

2. Can the expression "slant height" be applied to a pyramid if the
altitudes of its faces are not all equal? Discuss fully.

LATERAL AREA OF A REGULAR PYRAMID

237. THEOREM I. *The lateral area of a regular pyramid is equal to one half the product of its slant height and the perimeter of the base.*
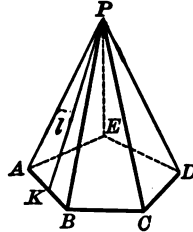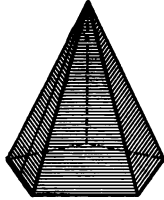
Suggestion for Proof: Let $l = PK$ represent the slant height, S the lateral area, and p the perimeter of the base. We are to show that

$$S = \tfrac{1}{2} l \cdot p.$$

Give all the steps.

238. Pyramids Cut by a Plane. The part of a pyramid between its base and a plane cutting all its edges is called a *frustum* if the cutting plane is parallel to the base, and a *truncated pyramid* if the cutting plane is not parallel to the base.
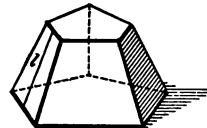
239. The Parts of a Frustum. The two parallel faces of a frustum are its *upper and lower bases,* the perpendicular distance between the bases is its *altitude,* the trapezoidal faces are its *lateral surface,* and the common altitude of these faces is the *slant height* of the frustum.
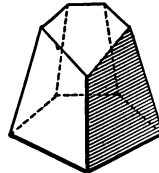
240. COROLLARY. *The lateral area of the frustum of a right pyramid is equal to half the sum of the perimeters of the bases multiplied by the slant height.*

Suggestion. Use § 62 and the figure in § 238.

SECTION PARALLEL TO THE BASE OF A PYRAMID

241. THEOREM II. *If a pyramid is cut by a plane parallel to the base, then*

(1) *the edges and altitude are divided proportionally;*

(2) *the polygonal section is similar to the base ;*

(3) *the areas of this section and of the base are in the same ratio as the squares of their perpendicular distances from the vertex.*
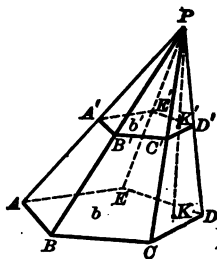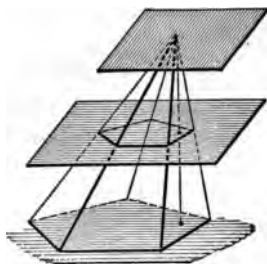
Outline of Proof : Let $A'B'C'D'E'$ be a section $\parallel ABCDE$.

(1) To prove that $\dfrac{PK'}{PK} = \dfrac{PA'}{PA} = \dfrac{PB'}{PB}$, etc., pass another plane through P parallel to the base and then use § 103.

(2) To prove that $ABCDE \sim A'B'C'D'E'$, we show that $\angle A = \angle A'$, $\angle B = \angle B'$, etc., and also $\dfrac{A'B'}{AB} = \dfrac{B'C'}{BC}$, etc.

(3) Calling the area of the section b' and that of the base b, we are to prove that $\dfrac{b'}{b} = \dfrac{\overline{PK'}^2}{\overline{PK}^2}$, and for this we need to show that

$$\frac{A'B'}{AB} = \frac{PA'}{PA} = \frac{PK'}{PK}.$$

Give all the steps in detail.

242. COROLLARY. *If two pyramids have equal altitudes and bases of equal areas lying in the same plane, the sections made by a plane parallel to the plane of the bases have equal areas.*
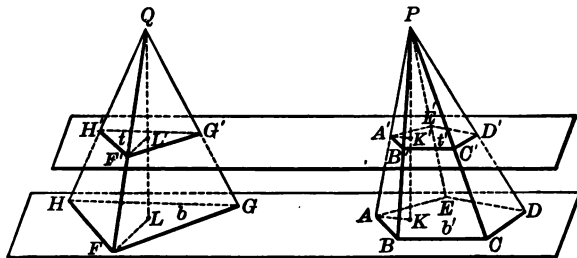
Suggestion. If t and t' are the areas of the sections and b and b' those of the bases, show that $\dfrac{t}{b} = \dfrac{\overline{QL'}^2}{\overline{QL}^2}$, $\dfrac{t'}{b'} = \dfrac{\overline{PK'}^2}{\overline{PK}^2}$, and hence $\dfrac{t}{b} = \dfrac{t'}{b'}$, from which it follows that $t = t'$ if $b = b'$.

SIGHT WORK

1. Find the lateral area of a regular pyramid if its slant height is 10 in. and the perimeter of its base is 25 in.

2. What is the perimeter of the base of a regular pyramid if its lateral area is 72 sq. in. and its slant height is 8 in. ?

3. What is the slant height of a regular pyramid if its lateral area is 160 sq. in. and the perimeter of its base is 20 in. ?

4. What is the perimeter of the base of a regular pyramid if its lateral area is 250 sq. in. and its slant height is 20 in. ?

5. The area of the base of a pyramid is 36 sq. in. and its altitude is 8 in. What is the area of a section of this pyramid made by a plane parallel to the base and 4 in. from the vertex ?

6. The area of the base of a pyramid is 64 sq. in. and its altitude is 8 in. Find the distance from the vertex to a plane parallel to the base if the area of its section is 16 sq. in.

7. If the area of one section of a pyramid is double that of another, both sections being parallel to the base, find the ratio of their distances from the vertex. Find this ratio if the area of one section is three times that of the other ; also if it is four times.

INSCRIBED AND CIRCUMSCRIBED PRISMS

243. Construction. A triangular pyramid is cut by a series of planes parallel to the base, including one through the vertex and also the one in which the base lies.

Through the lines of intersection of these planes with one of the faces, planes are constructed parallel to the opposite edge, thus forming a set of prisms all lying within the pyramid, as a', b', c' in pyramid P', or a set lying partly outside the pyramid, as a, b, c, d in pyramid P.
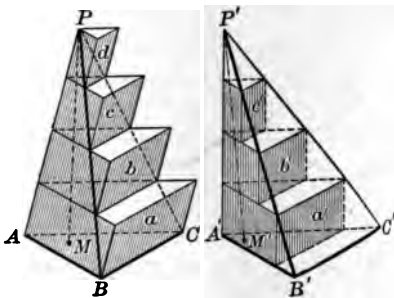
The inner prisms thus constructed are called *a set of inscribed prisms*, and the outer prisms are called *a set of circumscribed prisms*.

This process may be repeated by doubling the number of planes drawn parallel to the base and thus doubling the number of inscribed or circumscribed prisms. By continuing in this way either set of prisms may be made to coincide as nearly as we please with the pyramid.

244. Fundamental Assumption on the Volume of a Pyramid. We now assume that

A pyramid has a definite volume which is less than the combined volume of any set of circumscribed prisms and greater than that of any set of inscribed prisms.

SIGHT WORK

1. Show that in the above figure prisms b and a' are equal in volume if $PM = P'M'$ and $\triangle ABC$ and $A'B'C'$ have equal areas. See §§ 188, 242. Also prove that volume $c =$ volume b', etc.

2. Show that the sum of the volumes of the circumscribed prisms exceeds the sum of the volumes of the inscribed prisms by the volume of a.

PYRAMIDS HAVING EQUAL VOLUMES

245. THEOREM III. *If two triangular pyramids have equal altitudes and bases of equal areas, their volumes are equal.*
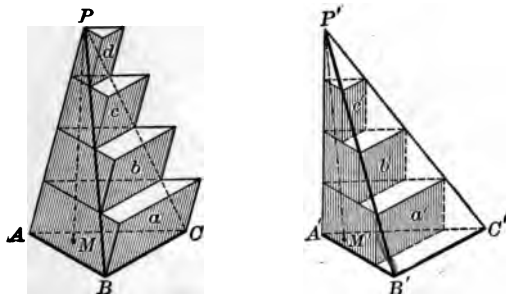
Given the pyramids P and P' in which altitudes PM and $P'M'$ are equal, and the bases ABC and $A'B'C'$ have equal areas.

To prove that P and P' have equal volumes.

Proof: If P differs at all in volume from P', let P be the greater, and let the difference be some fixed number K, so that
$$\text{Vol. } P - \text{Vol. } P' = K. \tag{1}$$

Construct a set of inscribed and circumscribed prisms as in §243.

Then $\qquad a' = b, \; b' = c, \; c' = d.$ Why?

Denote $\quad a + b + c + d$ by V and $a' + b' + c'$ by V'.

Then $\qquad\qquad V - V' = a. \tag{2}$

We have Vol. $P < V$, and Vol. $P' > V'$. §244

$\qquad\qquad \therefore \text{Vol. } P - \text{Vol. } P' < V - V'.$ Why?

Hence from (2) $\quad \text{Vol. } P - \text{Vol. } P' < a.$

Now take the divisions on PM small enough to make $a < K$. §243

Hence, $\qquad \text{Vol. } P - \text{Vol. } P' < K. \tag{3}$

Thus (3) contradicts (1), and hence the supposition that P and P' differ in volume is impossible. Q. E. D.

246. THEOREM IV.   *The volume of a triangular pyramid is one third of the product of its base and altitude.*
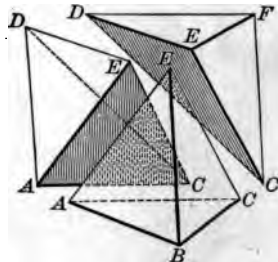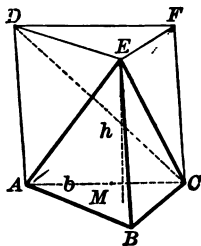
Given the triangular pyramid *E–ABC*. Let *h*, *b*, and *V* be the numerical measures respectively of the altitude *EM*, the area of the base *ABC*, and the volume of the pyramid.

To prove that $V = \frac{1}{3} bh.$

Proof: On the base *ABC* construct a triangular prism with altitude *h* and lateral edge *EB*.

This prism may be cut into three pyramids by the plane sections through *DEC* and *AEC*, as shown in the figure to the right.

The pyramids *E–ABC* and *C–DEF* have equal volumes since they have equal bases, *ABC* and *DEF* (§ 175), and the same altitude, *EM*.

Likewise volume *E–ACD* = volume *E–CFD*.

But *C–DEF* and *E–CFD* are the same pyramid.

Hence, Vol. *E–ABC* = Vol. *C–DEF* = Vol. *E–ACD*.

That is, Vol. *E–ABC* is one third of the volume of the prism.

But   volume of prism = *bh*.   § 198

Hence, *V* = volume of pyramid = $\frac{1}{3}$ *bh*.   Q. E. D.

VOLUME OF ANY PYRAMID

247. THEOREM V. *The volume of any pyramid is one third of the product of its base and altitude.*
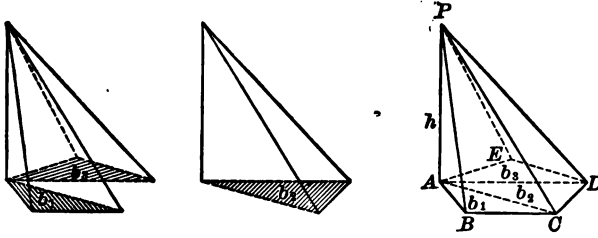
Given the pyramid $P\text{-}ABCDE$. **Let** V, b, **and** h **be the numerical measures respectively of the volume, base, and altitude.**

To prove that $V = \frac{1}{3} bh.$

Proof: By means of diagonal planes such as PAC and PAD, divide the given pyramid into triangular pyramids. Call the bases of the triangular pyramids b_1, b_2, b_3, etc., and their common altitude h. Then by § 246, the volumes of the triangular pyramids are $\frac{1}{3} b_1 h, \ \frac{1}{3} b_2 h, \ \frac{1}{3} b_3 h.$

Hence the total volume is $\frac{1}{3}(b_1 + b_2 + b_3)h = \frac{1}{3} bh.$ Q. E. D.

248. COROLLARY 1. *The volumes of any two pyramids having equal altitudes are proportional to the areas of their bases.*
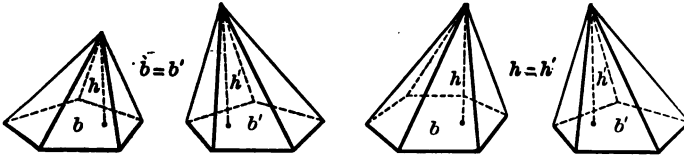
249. COROLLARY 2. *The volumes of any two pyramids having bases of equal areas are proportional to their altitudes.*

VOLUME OF A FRUSTUM OF A PYRAMID

250. THEOREM VI. *The volume of a frustum of a pyramid is equal to the combined volumes of three pyramids whose common altitude is the same as that of the frustum, and the areas of whose bases are those of the upper and lower bases of the frustum and the mean proportional between these areas.*
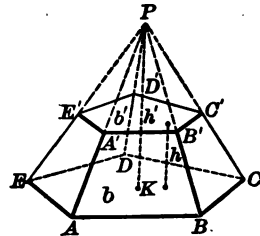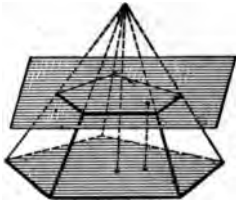
Given the frustum AC' with area of lower base b, area of upper base b', and altitude h. Then $\sqrt{bb'}$ is the mean proportional between b and b'.

To prove that the volume of AC' is

$$V = \tfrac{1}{3}hb + \tfrac{1}{3}hb' + \tfrac{1}{3}h\sqrt{bb'} = \tfrac{1}{3}h[b + b' + \sqrt{bb'}].$$

Proof: Let h' be the altitude PK of the complete pyramid $P\text{-}ABCDE$. Then the altitude of the pyramid $P\text{-}A'B'C'D'E'$ is $h' - h$.

Hence,
$$\frac{b}{b'} = \frac{h'^2}{(h' - h)^2} \qquad\qquad \text{§ 241 (3)}$$

from which
$$\frac{\sqrt{b}}{\sqrt{b'}} = \frac{h'}{h' - h}.$$

Clearing of fractions and solving for h',

$$h' = \frac{h\sqrt{b}}{\sqrt{b} - \sqrt{b'}} \qquad\qquad (1)$$

Now V is the difference between the pyramids whose .titudes are h' and $h' - h$.

Hence, $\qquad V = \frac{1}{3} bh' - \frac{1}{3} b'(h' - h),$ § 247

r, rearranging, $\qquad V = \frac{1}{3} b'h + \frac{1}{3} h'(b - b').$ (2)

Substituting the value of h' from (1) in (2) we have

$$V = \frac{1}{3} b'h + \frac{1}{3} \frac{h\sqrt{b}}{\sqrt{b} - \sqrt{b'}}(b - b')$$

$$= \frac{1}{3} b'h + \frac{1}{3} \frac{h\sqrt{b}}{\sqrt{b} - \sqrt{b'}}(\sqrt{b} - \sqrt{b'})(\sqrt{b} + \sqrt{b'})$$

$$= \frac{1}{3} b'h + \frac{1}{3} hb + \frac{1}{3} h\sqrt{bb'}$$

$$= \frac{1}{3} h\,[b' + b + \sqrt{bb'}].$$ Q. E. D.

SIGHT WORK

1. A flower bed is in the form of a regular right pyramid, with a square base 5 ft. on a side. The altitude is 3 ft. Find the number of cubic feet of soil used in its construction.

2. The altitude of a certain pyramid is 15 in. and its volume is 880 cu. in. Find the area of its base.

3. The area of the base of a pyramid is 48 sq. ft. and its volume 160 cu. ft. Find its altitude.

4. Find the locus of the vertices of pyramids having the same base and equal volumes.

5. Two monuments having bases of equal areas are pyramidal in shape, one being 15 ft. high and the other 18 ft. Find the ratio of their volumes.

6. Two pyramids with equal altitudes have bases whose areas are 7 sq. ft. and 13 sq. ft. Find the ratio of the volumes of the pyramids.

7. Find the volume of a frustum of a pyramid the areas of whose bases are 36 sq. in. and 144 sq. in., and whose altitude is 10 in.

8. The volume of a frustum of a pyramid is 332 cu. in. and the areas of its bases are 9 sq. in. and 36 sq. in. Find its altitude.

EXERCISES INVOLVING NUMERICAL COMPUTATION

1. The base of a regular pyramid is a square whose sides are 16 ft. Find the slant height of the pyramid if its altitude is 6 ft. Also find the lateral area.

2. The lateral area of a regular hexagonal pyramid is 72 sq. ft. and the slant height is 12 ft. Find the perimeter of the base, the apothem of the base, and the altitude of the pyramid.

3. Find the lateral area of the frustum of a pyramid if the perimeters of its bases are 27 and 54 in. respectively and its slant height is 12 in. Does the result depend upon the number of sides of the frustum?

4. The bases of a frustum of a pyramid are squares whose sides are 8 in. and 2 in. respectively. Find the volume of the frustum if its altitude is 6 in.

5. Find the volume of a regular triangular pyramid the sides of whose base are 5 in. and whose altitude is 6 in.

Suggestion. The area of a regular triangle with side a is $\dfrac{a^2}{4}\sqrt{3}$.

6. Find the volume of a regular hexagonal pyramid the sides of whose base are 8 in. and whose altitude is 10 in.

Suggestion. The area of a regular hexagon with side a is $\dfrac{3\,a^2}{2}\sqrt{3}$.

7. Two marble ornaments of equal altitudes are pyramidal in form. One has a square base 2 in. on a side and the other a regular hexagonal base 1 in. on a side. Find the ratio of their volumes.

8. A pyramid has for its base a right triangle with hypotenuse 10 and shortest side 6. Find the volume of the pyramid if its altitude is 9.

9. The slant height of a frustum of a regular pyramid is 10 in. and the apothems of its bases are 8 in. and 6 in. respectively. Find its altitude.

EXERCISES INVOLVING ALGEBRAIC COMPUTATION

1. A tent is to be made in the form of a right pyramid with a regular hexagonal base. If the altitude is fixed at 12 ft., what must be the side of the base in order that the tent may inclose 400 cu. ft. of space?

2. A pyramid with altitude 8 in. and a base whose area is 36 sq. in. is cut by a plane parallel to the base so that the area of the section is 18 sq. in. Find the distance from the base to the cutting plane.

3. A frustum of a pyramid is cut from a pyramid the perimeter of whose base is 60 in. and whose altitude is 15 in. What is the altitude of the frustum if the perimeter of its upper base is 20 in.? Does the result depend upon the number of sides of the pyramid?

4. Solve the preceding problem if the perimeter of the upper base of the frustum is one nth that of the lower base.

5. The area of the base of a pyramid is 180 sq. in. and its altitude is 20 in. Cut from it a frustum the area of whose upper base is 45 sq. in.; also one the area of whose upper base is one nth of 180 sq. in. Do these results depend upon the number of sides of the pyramid?

6. If the altitude of a pyramid is h, how far from the base must a plane parallel to it be drawn so that the area of its cross-section shall be half that of the base of the pyramid?

7. In a regular right pyramid a plane parallel to the base cuts it so as to make a section whose area is one half that of the base. Find the ratio between the lateral area of the pyramid and that of the small pyramid cut off by the plane.

8. Find the volume and the total surface of a regular tetrahedron whose edges are 9 in.

9. Find the total surface and the volume of a regular hexagonal pyramid the sides of whose base are each a and whose altitude is a.

CONES

251. Conical Surface. Given a closed convex
plane curve and a fixed point P not in its
plane. If a line through P moves so as
always to touch the curve and is made to
traverse it completely, it is said to generate
a *convex conical surface.*

The moving line is the *generator* of the sur-
face, and in any of its positions it is an *element*
of the surface. The fixed point is the *vertex.*
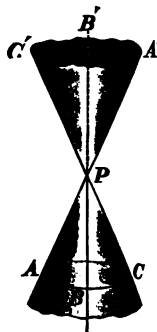
252. Nappes. A conical surface has two
parts, one on each side of the vertex, which are called *nappes.*

253. Cone. The solid bounded by a conical surface and a
plane section cutting all its elements, and
not passing through the vertex, is called a
cone.

254. Base. Lateral Surface. Altitude. The
plane part of the surface of a cone is its
base and the curved part is the *lateral sur-
face.* The *altitude* of a cone is the perpen-
dicular distance from the vertex to the plane
of the base.
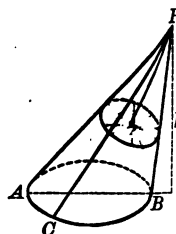
255. Circular Cone. A cone which has a circular cross-section
such that the perpendicular upon it from the vertex meets it
at the center is called a *circular cone.* If the base is such a
circle, the cone is then a *right circular cone.* Otherwise, it is
an *oblique circular cone.*

The *axis* of a circular cone is the line from the vertex
through the center of a circular section.

SIGHT WORK

If a plane cuts both nappes of a conical surface, show that it cannot
cut all the elements of the surface. See *suggestion* on page 79.

256. Generating a Right Circular Cone. A right circular cone may be *generated* by rotating a right triangle *PMB* about one of its legs, *PM*, as an axis. The hypotenuse *PB* generates the lateral surface, and the other leg, *MB*, generates the base.
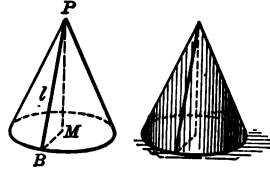
257. Slant Height. The generator of the convex surface of a right circular cone in any of its positions is called the *slant height* of the cone.

258. Frustum of a Cone. The part of a cone included between the base and a plane section parallel to the base is called a *frustum of a cone*.

The base of the cone and the parallel section are the *bases of the frustum*.

259. Slant Height of a Frustum. A frustum cut from a right circular cone has two circular bases. The segments of all the elements of the cone intercepted between these bases are equal and their common length is the *slant height* of the frustum.
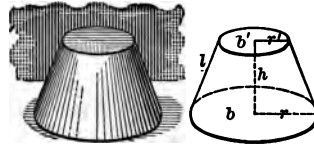
SIGHT WORK

1. Compare the definitions of pyramidal and conical surfaces.

2. Compare the definitions of a pyramid and a cone, and of a frustum of each.

3. What kind of cone corresponds to a regular pyramid?

4. Can the expression "slant height" be applied to any other cone than a right circular cone? Discuss this question for the frustum of a cone.

5. In a right circular cone, if any two of the three quantities, altitude, slant height, radius of the base, are given, show that the third may be found.

SECTION OF A CONE THROUGH AN ELEMENT

260. THEOREM VII. *If a plane contains an element of a cone and meets its surface in any other point, then it contains another element also, and the section is a triangle.*
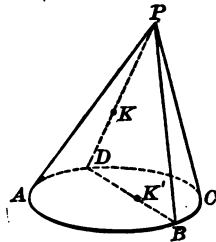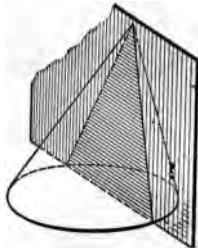
Let a plane contain the element *PB* of the cone *P–ABC*, and also one other point *K* or *K'*.

To prove that this plane contains another element PD, and that the section is a triangle PBD.

Outline of Proof : (1) *When the point K is in the lateral surface.*

Draw the element *PD* through *K*, and let the plane determined by *PB* and *PD* meet the base in *BD*.

Then △ *PBD* is the section made by the plane containing *PB* and the point *K*.

(2) *When the point K' is in the base.*

Draw *BD* through *K'* and also the element *PD*.

Then the plane determined by *BD* and *PD* contains *PB* and *K'* and cuts the cone in the △ *PBD*.

261. Plane Tangent to a Cone. If a plane contains an element of a cone and no other point of the cone, the plane is *tangent* to the cone, and the element is called the *element of contact.*
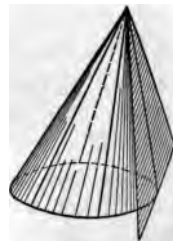
SECTION OF A CONE PARALLEL TO THE BASE

262. THEOREM VIII. *If the base of a cone is circular, every plane section parallel to the base is also circular.*
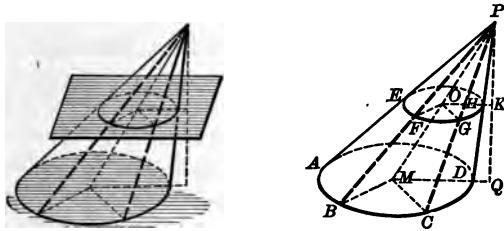
Given a cone with a circular base *AD.*

To prove that the ‖ section EH is also circular.

Proof: Draw the straight line from *P* to the center *M* of the base, and let it meet the section *EH* in the point *O.* Let *F* and *G* be any two points on the perimeter of the section *EH*.

Pass planes containing *PM* through the points *F* and *G*, and let them cut the base in *MB* and *MC* respectively.

Now in △ *PMB* and *PMC*, *OF* ‖ *MB* and *OG* ‖ *MC*. § 92

$$\therefore \frac{OF}{MB} = \frac{OP}{MP} = \frac{OG}{MC}$$

But   *MB = MC.* ∴ *OF = OG.*   Why?

Hence, since *F* and *G*, *any* two points on the perimeter of this section, are equally distant from *O*, this shows that *EH* is a circle whose center is *O.*   Q. E. D.

263. COROLLARY. *If a cone has a circular base, the areas of two cross-sections parallel to it are in the same ratio as the squares of their perpendicular distances from the vertex and also as the squares of the distances of their centers from the vertex.*

INSCRIBED AND CIRCUMSCRIBED PYRAMIDS

264. The lateral surface of a regular pyramid inscribed in a right circular cone may be computed and is equal to one half the product of the slant height and the perimeter of the base (§ 237).
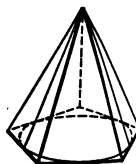
If the number of faces of the inscribed pyramid is doubled, the lateral surface of the resulting pyramid may again be computed in terms of its slant height and the perimeter of its base.
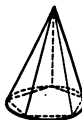
In a similar manner, the lateral surface of a regular pyramid circumscribed about a right circular cone may be computed in terms of the slant height of the cone and the perimeter of the polygon circumscribed about the base. The number of faces may be doubled and the lateral surface again computed, and so on.

Evidently either of these processes may be repeated indefinitely and the surfaces of the inscribed or circumscribed pyramids may be made to lie as close to the surface of the cone as we please.

The circumscribed pyramids all have the same slant height as that of the cone, and in case of the inscribed pyramids, the slant height may be made to differ by as little as we please from that of the cone by making the number of faces great enough.

265. Fundamental Assumption on the Lateral Area of a Right Circular Cone. We now assume that

A right circular cone has a definite lateral area which can be approximated as nearly as we please by taking the lateral area of the successive inscribed or circumscribed pyramids.

Since, by the theorem of § 237, the lateral area of any regular pyramid is half the product of the perimeter of its base and its slant height, it follows that this theorem holds for all pyramids used in approximating the lateral area of a right circular cone and that all practical measurements of such lateral areas are based on this theorem.
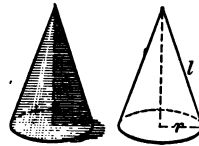
LATERAL AREA OF A RIGHT CIRCULAR CONE

The foregoing considerations constitute an informal proof of the following theorem:

266. THEOREM IX. *The area of the lateral surface of a right circular cone is equal to one half the product of its slant height and the circumference of its base.*

The argument used on page 96 shows that a theorem of this kind holds for every inscribed or circumscribed pyramid used in the approximation process, and hence this theorem for the cone is established for all purposes of practical measurement.

267. COROLLARY. *If l is the slant height of a right circular cone and r is the radius of its base, the area of the lateral surface is*

$$S = \tfrac{1}{2} \cdot 2\pi r \cdot l = \pi r l.$$

Note. In the case of a cone which is not a right circular cone the slant height varies from point to point and the process of computation of § 266 fails. Finding the lateral surface of such a cone depends on methods first introduced in the calculus and is a much more difficult problem than those solved in elementary plane and solid geometry.

SIGHT WORK

1. If a pyramid is inscribed in a cone, is its lateral area greater or less than that of the cone?

2. Does the theorem of § 237 apply to an irregular pyramid or to an oblique pyramid? Discuss fully.

3. If a regular pyramid of 1000 faces is inscribed in a right circular cone, would the theorem of § 237 apply to this pyramid? Would it be easy to detect experimentally the difference between this pyramid and the cone? Would the *size* of the cone affect the answer to this question? Discuss similar questions about a circumscribed pyramid of 1000 faces.

4. Find the lateral surface of a right circular cone whose altitude is 7 in. and the radius of whose base is 2 in.

LATERAL AREA OF A FRUSTUM

268. THEOREM X. *The lateral area of a frustum of a right circular cone is equal to one half the sum of the circumferences of the bases multiplied by the slant height.*
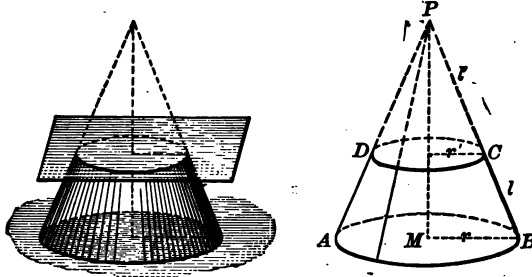
Given the frustum *ABCD*, with slant height l and radii of bases r and r'.

Let S represent its lateral area.

To prove that $S = \frac{1}{2}(2\pi r + 2\pi r')l = \pi l(r + r')$.

Proof: Complete the cone, and let $PC = l'$.

Then
$$S = \frac{1}{2}[2\pi r(l + l') - 2\pi r'l'] \qquad\qquad \text{§ 267}$$
$$= \pi rl + \pi l'(r - r'). \qquad\qquad (1)$$

But
$$\frac{r}{r'} = \frac{l + l'}{l'}, \text{ from which } l' = \frac{r'l}{r - r'}. \qquad (2)$$

Substituting l' from (2) in (1),
$$S = \pi rl + \pi r'l = \pi l(r + r'). \qquad\qquad \text{Q. E. D.}$$

269. COROLLARY. *The lateral area of a frustum of a right circular cone is equal to the circumference of a section midway between the bases multiplied by the slant height.*

Suggestion. In the formula of the theorem we have
$$S = \pi l(r + r') = 2\pi \frac{(r + r')}{2} l.$$

Now show that $\dfrac{r + r'}{2}$ is the radius of the section midway between the two bases.   § 46

SIGHT WORK

1. If a triangle which is not a right triangle revolves about one of its sides, does it generate a cone ?

Show that the figure generated by revolving any triangle about its longest side may be divided into two cones.

2. Find the lateral area of a right circular cone with radius of base 6 in. and slant height 10 in. In this and the succeeding exercises, express the results in terms of π, *e.g.*, 60π sq. in.

3. Find the lateral area of a right circular cone with radius of base 8 in. and altitude 6 in.

4. Find the lateral area of a frustum of a right circular cone, the radii of whose bases are 8 in. and 4 in. and whose slant height is 6 in.

5. A right circular cone having a base with radius 6 ft. and altitude 8 ft. is cut by a plane parallel to its base and at a distance of 4 ft. from the vertex. Find the radius of this section. Also find its area.

EXERCISES

1. The lateral area of the surface of a right circular cone is 120π sq. in., and its radius is 4 in. Find its slant height.

2. A circular chimney 100 ft. high is in the form of a frustum of a right circular cone whose lower base is 10 ft. in diameter and upper base 8 ft. Find the lateral surface.

3. The lateral area of a frustum of a right circular cone is 60π sq. in.; the radii of the two bases are 6 in. and 4 in. Find the slant height of the frustum.

4. The lateral area of a right circular cone is S, and the slant height is l. Express the radius of the base and also the altitude in terms of S and l.

5. If the radius of the base of a right circular cone is r, and the lateral area is S, express the slant height in terms of r and S.

6. If the slant height of a right circular cone is l, and the lateral area is S, express the circumference of the base in terms of l and S.

VOLUME OF A CONE

270. Consider inscribed and circumscribed pyramids similar to those used in § 264, except that they need not be regular since the cone is not now required to be a right circular cone.

By repeatedly increasing the number of faces of either the inscribed or circumscribed pyramids, they may be made to approach coincidence with the cone as nearly as we please.

271. Fundamental Assumption on the Volume of a Cone. We now assume that *a cone has a definite volume which can be approximated as nearly as we please by taking the volume of successive inscribed or circumscribed prisms.*

Recalling that the volume of a pyramid is equal to one third of the product of its base and altitude, we see that the above considerations constitute an informal proof of the following theorem:

272. THEOREM XI. *The volume of a cone is equal to one third of the product of its base and altitude.*

273. COROLLARY. *If the base of a cone is a circle with radius r and if the altitude is h, then the volume of the cone is*

$$V = \tfrac{1}{3} \cdot \pi r^2 \cdot h = \tfrac{1}{3} \pi r^2 h.$$
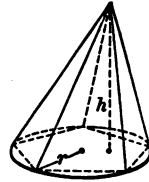
SIGHT WORK

1. Find the volume of an oblique cone with altitude 8 in. and a circular base whose radius is 6 in.

2. Find the volume of a right circular cone with slant height 10 in. and radius of base 6 in.

3. The area of the base of a cone is 50 sq. in. and its volume is 600 cu. in. Is the altitude the same whether the cone is right or oblique ?

4. Show that if two cones have bases of equal areas their volumes are proportional to their altitudes.

<div align="center">VOLUME OF A FRUSTUM OF A CONE</div>

274. THEOREM XII. *The volume of the frustum of a cone is equal to the combined volumes of three cones whose common altitude is the altitude of the frustum and whose bases are the upper and lower bases of the frustum and a mean proportional between these bases.*
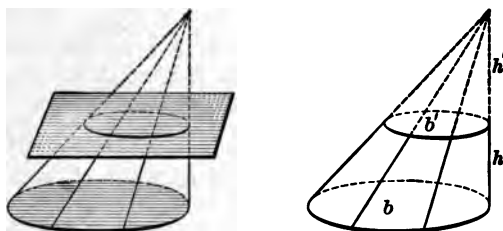
Suggestion. The proof is exactly like that of § 250, making use of § 272, instead of § 247. The result in symbols is $V = \frac{1}{3} h(b + b' + \sqrt{bb'})$.

275. COROLLARY. *The volume of a frustum of a right circular cone is $V = \frac{1}{3} \pi h(r^2 + r'^2 + rr')$, where h is the altitude and r and r' are the radii of the bases.*

For $b = \pi r^2$, $b' = \pi r'^2$. ∴ $V = \frac{1}{3} h(\pi r^2 + \pi r'^2 + \sqrt{\pi r^2 \cdot \pi r'^2})$
$= \frac{1}{3} \pi h(r^2 + r'^2 + rr')$.

Note. The theorem of § 272 holds for any cone whatever, whether right or oblique. However, when the base is not a circle we have no means in elementary mathematics for *computing* its area. Hence the volume of such a cone cannot be found at this stage even though § 272 does apply.

Similarly § 274 applies to a frustum of any cone whatever, but we are able to *compute* its volume by elementary methods only in case the bases are circles.

<div align="center">SIGHT WORK</div>

The radius of the base of a cone is 5 in. and its altitude is 10 in. Find the volume of a frustum formed by a plane parallel to the base and 6 in. from it. Find the total surface of this frustum in case the cone is a right circular cone.

SUMMARY OF BOOK IV

1. Define pyramidal surface and conical surface. In what respects do they differ?

2. Define pyramid, cone, regular pyramid, circular cone, and right circular cone.

3. For what kind of cones may the lateral area be found by means given in this Book? What is the corresponding kind of pyramid?

4. For what kind of frustum of a cone may the lateral area be found by means given in this Book? What is the corresponding kind of frustum of a pyramid?

5. What assumption about the volume of a pyramid is made in Book IV? In the proof of what theorems is this assumption used?

6. Beginning with the theorem of § 245 state in order the theorems which lead to the rule for finding the volume of any pyramid.

7. What assumption about the lateral area of a cone is made in this Book? Compare this assumption with the one in § 219.

8. What assumption is made about the volume of a cone? Compare this with the assumption in § 219.

9. What theorems on cylinders have no corresponding theorems for cones?

10. Show that a frustum of a cone becomes more and more nearly identical with a cylinder if the vertex of the cone is removed farther and farther from the base.

11. If a cone were regarded as a pyramid with a very large number of very narrow lateral faces, what pages in this Book would be superfluous?

12. Describe the difficulty in finding the lateral area of an oblique cone or of a cone whose base is not circular. Does the same difficulty exist in finding the volume of such a cone?

EXERCISES ON BOOK IV

1. If several planes are tangent to the same cone, find one point common to them all.

2. Find the locus of all lines which make a given angle with a given line at a given point in it.

3. Find the locus of all lines which make a given angle with a given plane at a given point.

4. If the middle points of four edges of a tetrahedron, no three of which meet at the same vertex, are joined, prove that a parallelogram is formed.
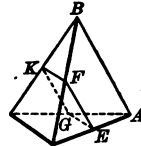
5. Show how to pass a plane through a tetrahedron so that the section shall be a parallelogram.
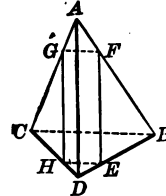
Suggestion. Pass a plane parallel to each of two opposite edges. See § 133.

6. A mound of earth of the shape shown in the figure has a rectangular base 16 yards long and 8 yards wide. Its perpendicular height is 5 yards, and the length on top is 8 yards. Find the number of cubic yards of earth in the mound.
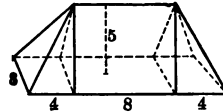
Suggestion. If from each end a pyramid with a base 8 yd. by 4 yd. is removed, the remaining part is a triangular prism.

7. Given a figure in general shape the same as the preceding, with a rectangular base of length 24 ft. and width 6 ft. Find its volume and lateral area if the dihedral angles around the base are each 45°.
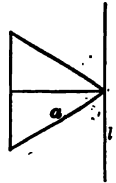
8. Find the area and volume of the figure developed by an equilateral triangle with sides a if it is revolved about one of its sides.

Suggestion. The figure may be divided into two cones.

9. Find the volume and area of the figure formed by revolving an equilateral triangle with sides s about an altitude.
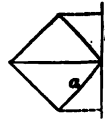
10. Find the area and volume of the figure developed by revolving a square whose side is a about one of its diagonals.

11. Through one vertex of an equilateral triangle with sides a draw a line l perpendicular to the altitude upon the opposite side. Find the volume and area of the figure developed by revolving the triangle about the line l.

Suggestion. The volume may be obtained by subtracting the volumes of two cones from the volume of a cylinder.

12. Through a vertex of a square with sides a draw a line l perpendicular to the diagonal at that point. Find the area and volume of the figure developed by turning the square around l.

Suggestion. Notice that two frustums and two cones are developed.

13. In a regular hexagon with sides a draw a line l bisecting two opposite sides. Find the area and volume of the figure developed by turning the hexagon about l as an axis.
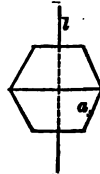
14. One angle of a right triangle is 30°. Find the ratios of the surfaces and also of the volumes of the solids developed by revolving this triangle around each of its three sides in succession.

Suggestion. The sides of the △ are a, $2a$, $a\sqrt{3}$.

15. If through any point P in a diagonal of a parallelopiped planes KN and RM are drawn parallel to two faces, show that the parallelopipeds DQ and LN thus formed have equal volumes.
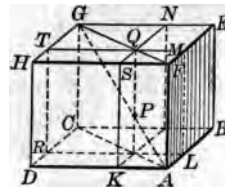
Archimedes (287–212 B.C.) was without doubt the greatest mathematician of his time. He is known not through any extended treatise, like that of Euclid, but through a series of monographs on practically every mathematical subject then known, including physics, mechanics, astronomy, and many phases of geometry. His work on the circle, cone, cylinder, and sphere is reflected in our treatment of surfaces and volumes at the present day.
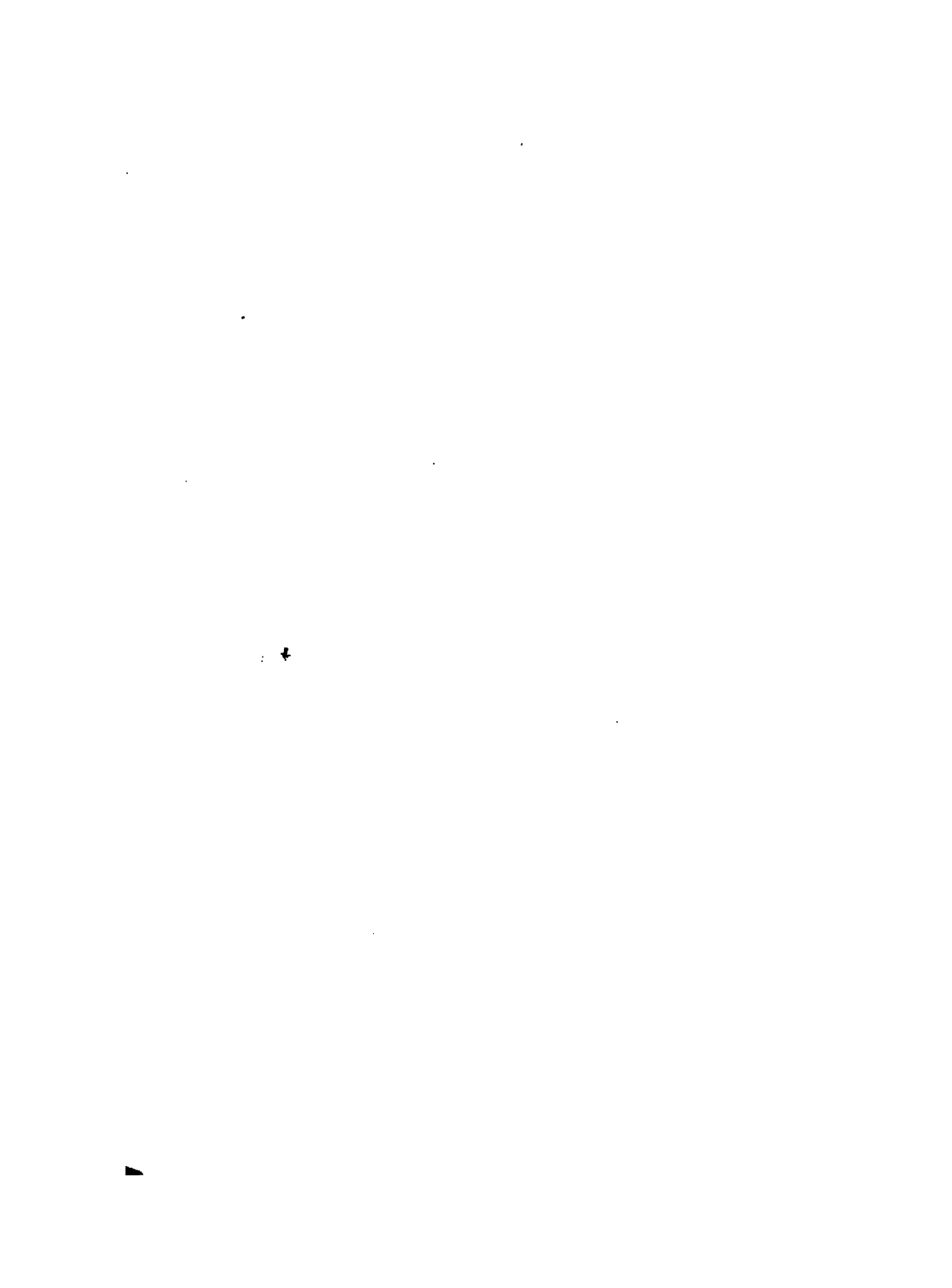
BOOK V

THE SPHERE

Sphere. Center. A *sphere* is a solid bounded by a sur-
points of which are equally distant from a point
called the *center*.

Diameter. Radius. A line-segment connecting two
on the surface of a sphere and passing through its
s a *diameter*. A segment joining the center to a point
surface is a *radius*.

Notation for a Sphere. A sphere
designated by a single letter at
ter or by two letters giving a
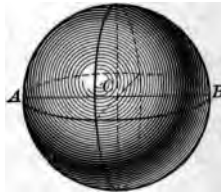
he sphere *C* means one whose center
the sphere *CA* is one whose center
whose radius is *CA*.

Generating a Sphere. The surface
here may be *generated* by revolv-
emicircle about its diameter as an
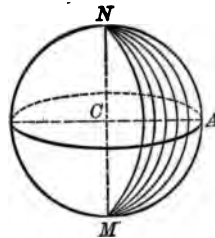
the surface of the sphere *CA* may be
by revolving the semicircle *MNA*
e diameter *MN*.

COROLLARY 1. *If two spheres have equal radii they may
to coincide and hence are equal.*

COROLLARY 2. *All radii of the same sphere or of equal
are equal.*

COROLLARY 3. *All diameters of the same sphere or of
heres are equal.*

105

283. THEOREM I. *A section of a spherical surface made by a plane is a circle.*

Given a sphere with center *C* cut by a plane *M*.

To prove that the points common to the surface of the sphere and the plane form a circle.

Proof: From the center *C* draw *CA* perpendicular to the plane *M*.

Let *B* and *D* be any two points common to the plane and the surface of the sphere. Complete the figure and prove *AB = AD*.
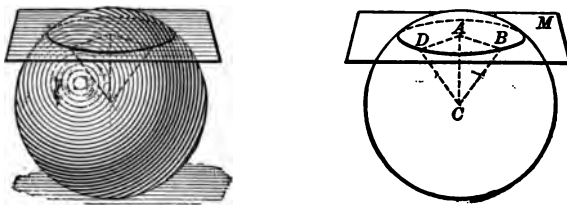
Hence any two points common to the surface of the sphere and the plane *M* are equidistant from the point *A*. That is, these points lie on a circle.

How must this proof be modified in case the plane *M* passes through the center of the sphere?

284. COROLLARY 1. *Through three points on a spherical surface there is one and only one circle.*

*Suggestion.* How many planes pass through these points?

285. COROLLARY 2. *A radius of a sphere through the center of a circle on its surface is perpendicular to the plane of the circle; and conversely, a radius of a sphere perpendicular to the plane of a circle on its surface passes through the center of the circle.*

PROPERTIES OF CIRCLES ON A SPHERE

286. Figures on a Sphere. Any figure drawn on the surface of a sphere is said to *lie on the sphere.*

287. Axis and Poles. The line perpendicular to the plane of a circle at its center is called the *axis* of the circle.

The points in which the axis of a circle on a sphere meets the surface of the sphere are called the *poles of the circle.*
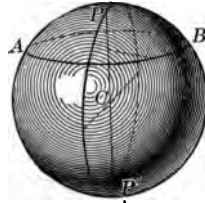
288. Great and Small Circles on a Sphere. If the plane of a circle on a sphere passes through the center of the sphere, it is called a *great circle* of the sphere, and if not, it is called a *small circle.*

Thus, in the figure, AB is a small circle, PP' is its axis, and P and P' are its poles.

The circle passing through P and P' and containing the center C is a great circle.

289. Inside and Outside of a Sphere. A point is inside, outside, or on a sphere according as its distance from the center is less than, greater than, or equal to the radius of the sphere.

290. COROLLARY 1. *The center of a great circle on a sphere is the center of the sphere.*

291. COROLLARY 2. *All great circles on a sphere are equal and any two such circles bisect each other.*

292. COROLLARY 3. *The axis of any circle on a sphere passes through the center of the sphere.*

293. COROLLARY 4. *Through two given points on a sphere there is one and only one great circle unless these points are at opposite ends of a diameter.*

294. COROLLARY 5. *Circles on a sphere formed by parallel planes have the same axis and the same poles.*

DISTANCES FROM POINTS ON A CIRCLE TO ITS POLE

295. Distance on a Sphere. The *spherical distance,* or simply the *distance* between two points on a sphere is the distance measured between these points along the minor arc of the great circle through them.

296. THEOREM II. *All points of a circle on a sphere are equidistant from either pole of the circle.*
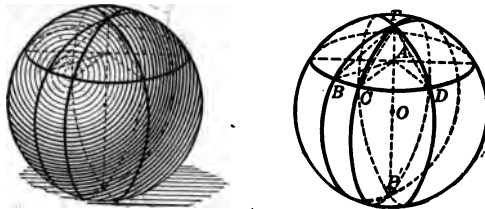
Given *P* a pole of the circle whose center is *A*, with *B, C,* and *D* any points on this circle.

To prove that the great circle arcs PB, PC, and PD are equal.

Suggestion. Prove chord *PB* = chord *PC* = chord *PD*, and hence arc *PB* = arc *PC* = arc *PD*.   § 48

297. Polar Distance of a Circle. The spherical distance from the points of a circle on a sphere to its nearest pole is called the *polar distance* of the circle.

298. A Quadrant. One fourth of a great circle is a *quadrant.*

299. COROLLARY 1. *The polar distance of a great circle is a quadrant.*

300. COROLLARY 2. *If a point on a sphere is at a quadrant's distance from each of two points not at the extremities of the same diameter, it is the pole of the great circle through these points.*
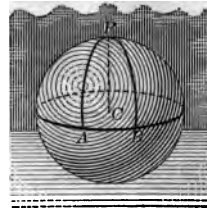
USE OF A SPHERICAL BLACKBOARD

301. COROLLARY 3. *Polar distances of equal circles on a sphere are equal.*

302. COROLLARY 4. *Straight line-segments joining points of a circle on a sphere to one of its poles are equal.*

It follows from the preceding theorem and corollaries that, if a *spherical* blackboard is at hand, circles may be constructed on it by means of crayon and string the same as on a *plane* blackboard. Likewise, curve-legged compasses may be used.

SIGHT WORK

1. If two points are at the extremities of the same diameter of a sphere, how many great circles can be passed through these points ?

2. What great circles on the earth's surface pass through both poles ? Where are the poles of these circles located ?

3. If A and B are at opposite ends of a diameter, can a small circle be passed through them ?

4. If two circles on a sphere have the same poles, prove that their planes are parallel. See § 294.

5. What is the locus of all points on a sphere at a quadrant's distance from a given point ?

6. What is the locus of all points on a sphere at any fixed distance from a given point on the sphere ? What is the greatest such distance possible ? Discuss fully.

7. If two planes cutting a sphere are parallel, what can be said of the centers of the circles thus formed ? What can be said of the poles of these circles ? Are these statements true of three or more such circles ?

8. Find the locus of the centers of a set of circles on a sphere formed by a set of parallel planes cutting it.

9. AB is a fixed diameter of a sphere. A plane containing AB is made to revolve about it as an axis. Find the locus of the poles of the great circles on the sphere made by this revolving plane. How are the points A and B related to this locus ?

303. THEOREM III. *If the planes of two circles on a sphere are equidistant from the center, the circles are equal; and conversely, if two circles on a sphere are equal, their planes are equidistant from the center.*
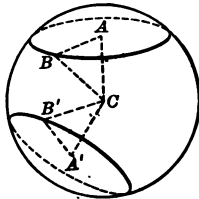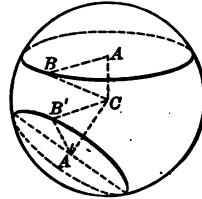
FIG. 1   FIG. 2

Suggestion. In the first figure show that (1) if $CA = CA'$, then $AB = A'B'$, and (2) if $AB = A'B'$, then $CA = CA'$.

CIRCLES UNEQUALLY DISTANT FROM THE CENTER

304. THEOREM IV. *If the planes of two circles on a sphere are unequally distant from the center, the circles are unequal, the one nearer the center being the greater; and conversely, if two circles on a sphere are unequal, the plane of the greater circle is nearer the center.*

Suggestion. In Fig. 2 above, show (1) that if $CA < CA'$ then $AB > A'B'$; and (2) if $AB > A'B'$ then $CA < CA'$.

305. Plane Tangent to a Sphere. A plane which meets a sphere in only one point is *tangent to the sphere.*

Two spheres are *tangent to each other* if they are both tangent to the same plane at the same point.

A *line is tangent to a sphere* if it contains one and only one point of the sphere.

PLANE TANGENT TO A SPHERE

306. THEOREM V. *A plane tangent to a sphere is perpendicular to the radius from the point of tangency; and conversely, a plane perpendicular to a radius at its extremity is tangent to the sphere.*
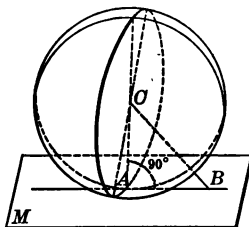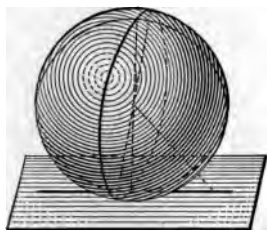

Given a sphere C **with plane** M **tangent to it at** A.

To prove that the radius CA is perpendicular to the plane M.

Proof: (1) It is necessary to prove that CA is perpendicular to every line in M through A. (Why?)

Draw any such line as AB.

The plane BAC cuts the sphere in a circle. Then AB is tangent to this circle, and hence perpendicular to AC.

Since CA is $\perp$ to every line in M through A, it is $\perp$ to M.

(2) To prove the converse, note that CA is the shortest distance from C to the plane M. § 82

Hence, every point of M except A is *exterior* to the sphere.

That is, M is a tangent plane. (§ 305.) <small>Q. E. D.</small>

**307. Sphere Inscribed in a Polyhedron.** A sphere is said to be *inscribed in a polyhedron* if every face of the polyhedron is tangent to the sphere. The polyhedron is then said to be *circumscribed about the sphere.*

308. Polyhedron Inscribed in a Sphere. A polyhedron is *inscribed in a sphere* if all its vertices lie in the surface of the sphere. The sphere is then *circumscribed about the polyhedron.*

SPHERE INSCRIBED IN A TETRAHEDRON

309. PROBLEM. *To inscribe a sphere in a given tetrahedron.*
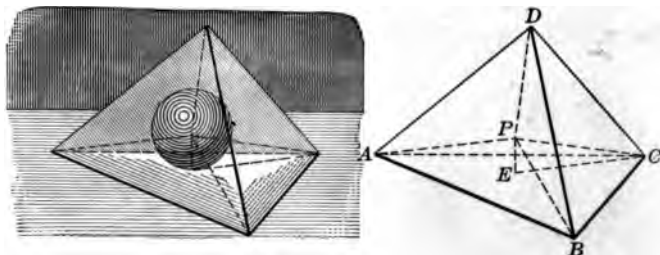
Given a tetrahedron *D–ABC*.

To construct a sphere tangent to each of its four faces.

Construction: Construct planes bisecting the dihedral angles whose edges are *AB, BC,* and *CA.* These three planes meet in a point *P,* which is equally distant from the four faces of the tetrahedron. With the point *P* as a center and a radius *PE* equal to the distance from *P* to one of the faces construct a sphere. This sphere is inscribed in the tetrahedron.

Proof: Every point in the plane *PAB* is equidistant from the faces *ABD* and *ABC.*   § 123

Similarly every point in the plane *PBC* is equidistant from the planes *ABC* and *DBC,* and every point in the plane *PAC* is equidistant from the planes *ABC* and *DAC.*

Hence, the point *P,* common to all three planes, is equidistant from each of the four faces of the tetrahedron.

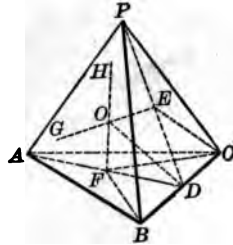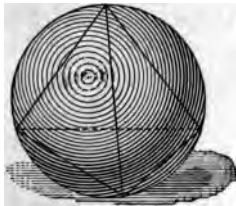∴ each plane is tangent to the sphere, and the sphere is inscribed in the tetrahedron.   (§ 307.)   Q. E. F.

SIGHT WORK

Discuss the problem of finding a point equally distant from **three planes** two of which are parallel. How many such points are there?

SPHERE CIRCUMSCRIBED ABOUT A TETRAHEDRON

310. PROBLEM. *To find a point equally distant from the four vertices of a tetrahedron.*

Given the tetrahedron *P–ABC*.

To find a point O equidistant from P, A, B, C.

Construction : At *D*, the middle point of *BC*, construct a plane perpendicular to *BC*.

This plane contains *E*, the center of the circle circumscribed about △ *PBC* and also *F*, the center of the circle circumscribed about △ *ABC*.   Why ?

In the plane *FDE* draw *EG* ⊥ *ED* and *FH* ⊥ *FD*.

Then *EG* and *FH* cannot be parallel and hence meet in some point *O*.   Then *O* is the point required.

Proof : Since *BC* is ⊥ to the plane *FDE*, it follows that each of the planes *PBC* and *ABC* is ⊥ to plane *FDE*.   § 117

Hence *OE* is ⊥ plane *PBC* and *OF* is ⊥ plane *ABC*.   § 114

Then *O* is equidistant from *P*, *B*, and *C*, and also from *A*, *B*, and *C*.   § 81

Hence $OA = OB = OC = OP$.   Q. E. F.

311. COROLLARY 1.   *A sphere may be passed through any four points, not all of which lie in the same plane.*

312. COROLLARY 2.   *A sphere may be circumscribed about any tetrahedron.*

313. PROBLEM. *To find the diameter of a given material sphere.*

With any point P of the sphere as a pole, construct a circle, and on this circle select any three points A, B, C.

Using a pair of compasses, measure the straight line-segments AB, BC, CA, and construct the triangle $A'B'C'$ equal to ABC.

Let $B'D'$ be the radius of the circle circumscribed about $A'B'C'$.

If PP' is the axis of the circle ABC on the sphere and BD the radius of this circle, then $BD = B'D'$.
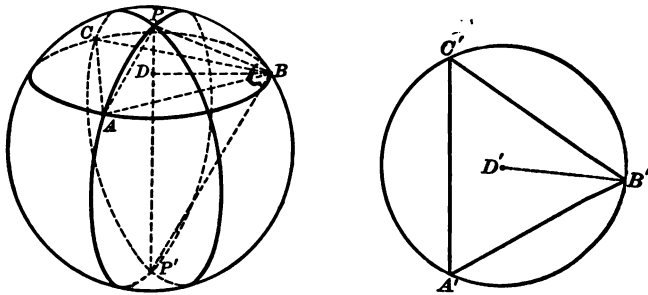
Measure PB by means of the compasses.

Then PBP' is a right triangle, with BD perpendicular to its hypotenuse PP'.

PB and BD being known, we may now *compute PD* from the right triangle PBD, and then compute PP' from the similar triangles PBD and $PP'B$, for the latter using the relation $PD : PB = PB : PP'$ or $PD \times PP' = \overline{PB}^2$.

The segment PP' may also be found by *geometric construction*; namely, by drawing a triangle equal to $P'BP$.

Show how to do this when BP and BD are known.

INTERSECTING SPHERICAL SURFACES

314. THEOREM VI. *The intersection of two spherical surfaces is a circle.*

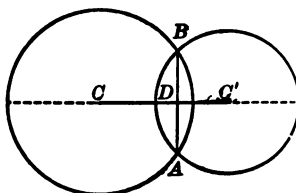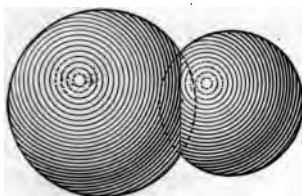**Proof:** Two intersecting spherical surfaces may be developed by two intersecting circles rotating about the line connecting their centers C and C'. Let A and B be the two points common to the two circles.

Then BA is $\perp CC'$.

$(CC'$ is the $\perp$ bisector of BA.) § 28

As the figure rotates about the line CC', AB remains $\perp CC'$ and therefore all its positions lie in a plane. § 79

Also DB and DA remain fixed in length.

Therefore the points A and B trace out a circle. Q. E. D.

SIGHT WORK

1. Find the locus of the centers of all spheres tangent to a given plane at a given point.

2. Find the locus of the centers of all spheres of given radius tangent to a given line at a given point.

3. Find the locus of the centers of all spheres of given radius tangent to a fixed plane.

4. Find the locus of the centers of all spheres of given radius tangent to a fixed line.

5. Find the locus of the centers of all spheres tangent to two given intersecting planes.

EXERCISES

1. The four lines perpendicular to the faces of a tetrahedron at their circumcenters meet in a point.

2. The six planes perpendicular to the edges of a tetrahedron at their middle points all meet in a point.

3. The planes bisecting the six dihedral angles of a tetrahedron all meet in a point.

4. Show that a sphere may be inscribed in a cube.

5. Show that a sphere may be circumscribed about a cube.

6. Can a sphere be circumscribed about a rectangular parallelopiped which is not a cube? Can a sphere be inscribed in it? Prove.

7. If a plane M is tangent to a sphere at a point A, show that the plane of every great circle of the sphere through A is perpendicular to M.

8. Show that the line of centers of two intersecting spheres meets the spherical surfaces in the poles of their common circle.

9. Show that two spheres are tangent if they meet on their line of centers. Distinguish two cases. State and prove the converse of this proposition.

10. If a sphere is tangent to a given plane M at a given point A, how many additional points on the sphere are required to determine it?

Suggestion. Suppose one point P given. Pass a plane through $P \perp$ to plane M at A. Is there only one such plane? Discuss fully.

11. Describe the set of all lines in space whose distances from the center of a sphere are all equal to the radius of the sphere.

12. Describe the set of all planes whose distances from the center of a sphere are all equal to the radius of the sphere.

13. Describe the set of all spheres of given radius which are tangent to a given sphere of greater radius.

SPHERICAL ANGLES

315. Spherical Angles. Two planes through the center of a sphere cut the surface of the sphere in two great circles which intersect in two points and form four *spherical angles* about each of these points. Any two of these angles are *adjacent* or *vertical* as in the case of angles formed by straight lines.
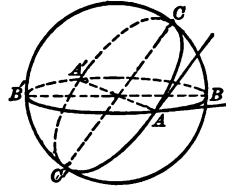
A spherical angle is measured by the angle between the tangents to its sides (arcs) at their common point.

Only angles formed by great circles are considered in this book.

MEASURING A SPHERICAL ANGLE
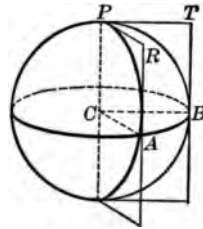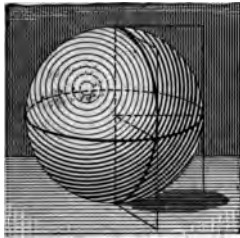
316. THEOREM VII. *A spherical angle is measured by an arc of the great circle whose pole is the vertex of the angle and which is intercepted by the sides of the angle.*

Suggestion. Show that $\overarc{AB}$ measures the dihedral angle formed by the planes *PAC* and *PBC* and that $\angle BCA = \angle TPR$.

317. COROLLARY 1. *The sum of the consecutive spherical angles about a point is four right angles.*

318. COROLLARY 2. *A spherical angle is equal to the dihedral angle formed by the planes of its arcs.*

SPHERICAL POLYGONS

319. Polygons and Triangles on a Sphere. The portion of a spherical surface contained within a polyhedral angle whose vertex is at the center of a sphere is called a *spherical polygon.*

It follows that for every spherical polygon there is a corresponding polyhedral angle at the center of the sphere made by drawing radii to the vertices of the polygon. The face angles of the polyhedral angle correspond to the sides of the spherical polygon and the dihedral angles of the polyhedral angle to the angles of the spherical polygon.

Since a plane through the center of a sphere intersects the surface in a great circle, it follows that the sides of a spherical polygon are *arcs of great circles.*

Since a plane may be passed through the vertex of a polyhedral angle such that the polyhedral angle lies entirely on one side of it, it follows that a spherical polygon *lies within one hemisphere.*

A spherical polygon of three sides is a *spherical triangle.*

320. Relation between the Parts of a Spherical Polygon and the Corresponding Polyhedral Angle.

(1) The face angles of the polyhedral angle are measured by the arcs forming the sides of the spherical polygon. Why?

(2) The dihedral angles of the polyhedral angle are equal in measure to the angles of the spherical polygon.   § 318

SIGHT WORK

Why is the vertex of a polyhedral angle which is used in defining a spherical polygon placed at the center of the sphere?
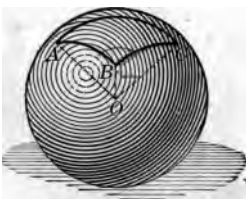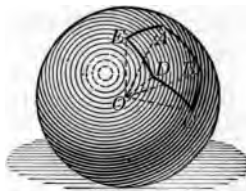
COROLLARIES ON SPHERICAL POLYGONS

The following propositions are now obvious corollaries of the preceding definitions and discussions.

321. *The sum of two sides of a spherical triangle is greater than the third side.*

This is a direct consequence of § 151.

322. *The sum of the sides of a spherical polygon is less than 360°.*

This is a direct consequence of § 152.

323. *Each side of a spherical polygon is less than 180°.*

SIGHT WORK

1. What is the angle between a meridian on the earth's surface and the equator ?

2. If two meridians are drawn meeting the equator 10° apart, what is the angle between these meridians ?

3. If two meridians meet the equator 90° apart, what can be said of the three angles of the spherical triangle thus formed ?

4. If in two spherical triangles the three sides of one are equal respectively to the three sides of the other, what can be said of the face angles of the corresponding trihedral angles ?

5. If in two spherical triangles the three angles of one are equal respectively to the three angles of the other, what parts are equal in the corresponding trihedral angles ?

6. If in two spherical polygons the angles of one are equal respectively to the angles of the other, what parts are equal in the corresponding polyhedral angles ?

7. If in two spherical polygons the sides of one are equal respectively to the sides of the other, what parts of the corresponding polyhedral angles are equal ?

8. Prove that if the sides of two spherical triangles are equal, then the angles of the triangles are equal.

Suggestion. Use § 144.

SHORTEST SPHERICAL DISTANCE BETWEEN TWO POINTS

324. Theorem VIII. *The shortest distance on a sphere between two of its points is measured along the minor arc of a great circle passing through these points.*

Proof: Let A and B be any two points on a sphere, AB the minor arc of a great circle through them, and $ADCB$ any other curve on the sphere connecting A and B. Let D and C be any two points on the curve $ADCB$ taken in order from A to B.
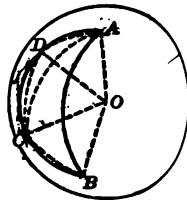
Draw the great circle arcs AD, AC, DC, and CB. Then by § 321 $\widehat{AC} + \widehat{CB} > \widehat{AB}$ and $\widehat{AD} + \widehat{DC} > \widehat{AC}$.

Hence, $\widehat{AD} + \widehat{DC} + \widehat{CB} > \widehat{AB}$.

If in like manner we subdivide AD, DC, CB, and continue this process, we obtain a succession of paths, each longer than the preceding, and thus we get closer and closer to the length of the curve $ADCB$.

Hence, the curve $ADCB$ must be greater than $\widehat{AB}$.   Q. E. D.

325. Symmetrical Spherical Triangles. Two spherical triangles are *symmetrical* if the sides and angles of one are equal respectively to the sides and angles of the other, but arranged in the *opposite* order. Compare § 146.
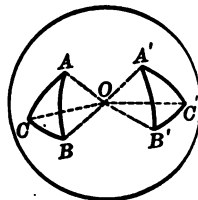
SIGHT WORK

1. Is it possible to move the spherical triangle ABC in the above figure so as to make it coincide with triangle $A'B'C'$?

2. If in two plane triangles ABC and $A'B'C'$ the corresponding parts are equal, is it always possible to make the triangles coincide? Is this always possible without inverting one of the triangles? Discuss the difference in this respect between plane and spherical triangles.

TRIHEDRAL ANGLES AND SPHERICAL TRIANGLES

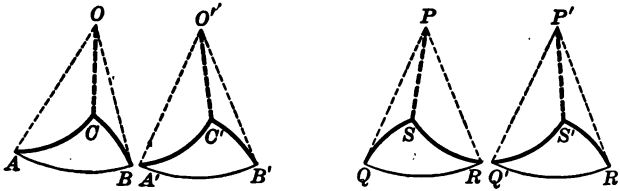**326. THEOREM IX.** *Two equal or symmetrical tri-hedral angles with vertices at the center of a sphere intercept equal or symmetrical triangles, respectively, on the sphere.*

Suggestions for Proof. In case the trihedral angles are equal they can be made to coincide, whereby the spherical triangles will also be made to coincide. The triangles are therefore equal. In case the trihedral angles are symmetrical it follows directly from §§ 146, 325 that the spherical triangles are symmetrical.

327. THEOREM X. *If the radii drawn from the vertices of a spherical triangle are extended, they meet the sphere in the vertices of a triangle symmetrical to the given triangle.*

The proof is left for the student.

328. THEOREM XI. *Two triangles on the same sphere, or on equal spheres, are equal or symmetrical if three sides of one are equal respectively to three sides of the other.*
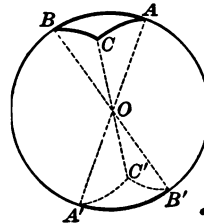
This is a direct corollary of §§ 145, 149. See the figure under § 326.

329. THEOREM XII. *Two triangles on the same sphere, or on equal spheres, are equal or symmetrical if two sides and the included angle of one are equal respectively to two sides and the included angle of the other.*

Proof: This is a direct corollary of the theorems §§ 142, 150.

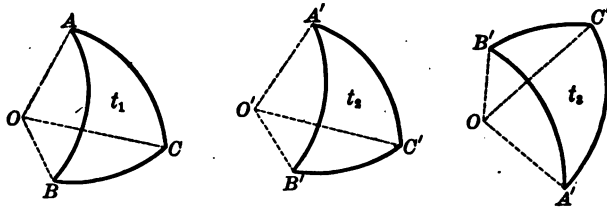**330. Vertical Spherical Angles.** Spherical angles are *vertical* if they have the same vertex and if the sides of one are extensions of the sides of the other.

331. Right Spherical Triangle. A *right* spherical triangle has one of its angles a right angle.

332. Isosceles Spherical Triangle. An *isosceles* spherical triangle has two equal sides.

1. Show that vertical spherical angles are equal.

2. What kind of spherical triangle is formed by two meridians on the earth's surface and the arc of the equator which they intercept?

3. Show that every spherical triangle formed as described in Example 2 contains more than two right angles.

4. Show how to form a spherical triangle as in Example 2 in which each angle shall be a right angle.

5. Show how to form a spherical triangle as in Example 2 in which one angle shall be 179° and each of the other two 90°.

EQUAL ANGLES OPPOSITE EQUAL SIDES

333. THEOREM XIII. *The angles opposite the equal sides of an isosceles spherical triangle are equal.*

Suggestion. Let AC and BC be the equal sides. Draw $\overset{\frown}{CD}$ to the middle point of $\overset{\frown}{AB}$. Then use § 328.
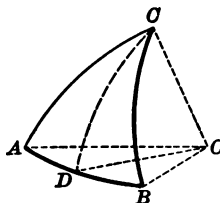
334. COROLLARY. *If two isosceles spherical triangles are symmetrical, they are equal, and conversely.*

EXERCISES

1. Compare fully the theorems on the equality of plane triangles and of trihedral angles. Is there any theorem in either case for which there is no corresponding theorem in the other?

2. Compare in the same manner the theorems on the equality of plane triangles and of spherical triangles.

3. Compare in the same manner the theorems on the equality of trihedral angles and of spherical triangles.

4. If two face angles of a trihedral angle are equal, the opposite dihedral angles are equal.

5. If two face angles of a trihedral angle are equal, it is equal to its symmetrical trihedral angle.

Suggestion. Compare the sides and then the angles of the corresponding spherical triangles. Use §§ 333, 329.

6. Show how to find a pole of the circle through three given points on a sphere.

Suggestion. Let the given points be A, B, C. By § 296 a pole of the circle is equidistant from A, B, and C. Connect A and B by an arc of a great circle and construct another arc of a great circle bisecting $\overset{\frown}{AB}$ perpendicularly. Similarly construct a perpendicular bisector of $\overset{\frown}{BC}$. The points in which these two arcs meet will be the poles of the circle through A, B, and C.

POLAR TRIANGLES

335. Definition. If with the vertices ·
of a given spherical triangle as poles
arcs of great circles are constructed,
another spherical triangle is formed which
is called the *polar triangle* of the first.

Thus in the figure, A is a pole of the arc
$B'C'$, B is a pole of the arc $C'A'$, and C is a
pole of the arc $A'B'$. Hence $A'B'C'$ is the
polar triangle of triangle ABC.
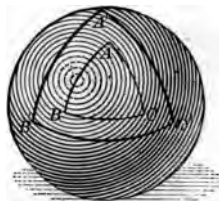
336. Polar Triangle. How Selected. If with the vertices
A, B, C of a spherical triangle as
poles three complete great circles
are constructed, each of these
circles meets each of the others
in two points, thus forming eight
spherical triangles, as shown
in the figure, namely, $A'B'C'$,
$A'B'F$, $B'C'D$, $C'A'E$, $A'EF$,
$B'DF$, $C'DE$, and DEF.

There is one and only one of
these, namely, $A'B'C'$, such that
A and A' are on the same side of circle $B'C'$, B and B' on the
same side of circle $A'C'$, and C and C' on the same side of
circle $A'B'$.

The triangle $A'B'C'$ as thus described is *the polar triangle*
of ABC.
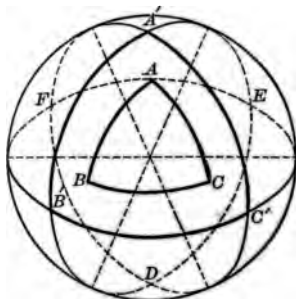
SIGHT WORK

In the above figure the parts of the great circles which are supposed
to be on the front side of the figure are given in solid lines while the
parts on the back side are dotted. Study the figure with care and state
which triangles are entirely on the front side, which are entirely on the
back side of the sphere, and which are partly on the front side and partly
on the back side of the sphere.

337. THEOREM XIV. *If $A'B'C'$ is the polar triangle of ABC, then ABC is the polar triangle of $A'B'C'$.*
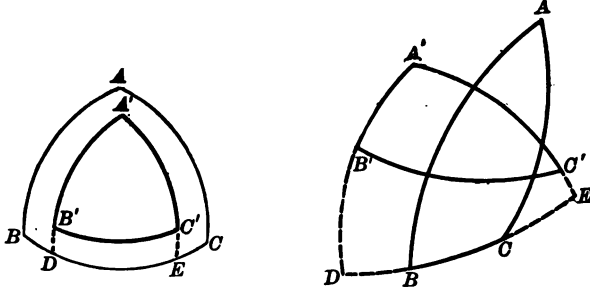
Given △ $A'B'C'$, the polar triangle of ABC.

To prove that ABC is the polar triangle of $A'B'C'$.

Proof: (1) A' is at a quadrant's distance from B because B is the pole of $\overset{\frown}{A'C'}$. A' is also at a quadrant's distance from C because C is the pole of $\overset{\frown}{A'B'}$.

Hence, A' is the pole of $\overset{\frown}{BC}$.   § 300

Similarly, B' is the pole of $\overset{\frown}{AC}$ and C' the pole of $\overset{\frown}{AB}$.

(2) To show that A and A' lie on the same side of the circle BC, we note that since A is the pole of the circle $B'C'$ and A lies on the same side of this circle with A', then A and A' are at less than a quadrant's distance. Hence, it follows that if A' is at a quadrant's distance from BC, A and A' must be on the same side of BC.

In like manner we show that B and B' lie on the same side of AC, and C and C' on the same side of AB.

338. Corresponding Parts of Polar Triangles. If ABC and $A'B'C'$ are polar triangles, and if A is a pole of $\overset{\frown}{B'C'}$, then $\angle A$ and $\overset{\frown}{B'C'}$ are said to be *corresponding parts*.

MEASURE OF PARTS IN POLAR TRIANGLES

339. THEOREM XV. *The sum of the measures of an angle of a spherical triangle and the corresponding arc of its polar triangle is* 180°.

Given the polar triangles *ABC* and *A'B'C'*. Denote the measures in degrees of the angles by *A, B, C, A', B', C'*, and of the corresponding sides by *a', b', c', a, b, c.*
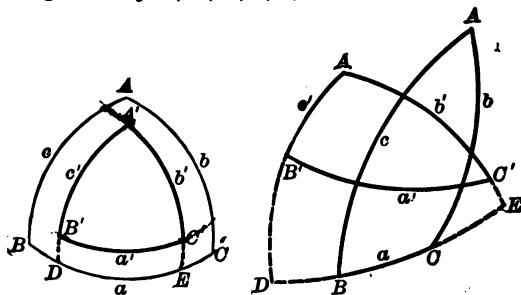
To prove that
$$A + a' = 180° \qquad A' + a = 180°$$
$$B + b' = 180° \qquad B' + b = 180°$$
$$C + c' = 180° \qquad C' + c = 180°$$

Suggestions for Proof: Extend (if necessary) arcs *A'B'* and *A'C'* till they meet the great circle *BC* in points *D* and *E*, respectively. Then arc *DE* is the measure of $\angle$ *A'*.

Also $\widehat{BE} = 90°$, and $\widehat{DC} = 90°$. (Why?)

But $\widehat{BE} + \widehat{DC} = \widehat{BC} + \widehat{ED} = a + A' = 180°$.

340. COROLLARY 1. *If in two spherical triangles an angle of one is equal to an angle of the other, then the corresponding sides of their polar triangles are equal; and conversely, if in two spherical triangles a side of one is equal to a side of the other, then the corresponding angles of their polar triangles are equal.*

341. COROLLARY 2. *If two spherical triangles are equal or symmetrical, their polar triangles are equal or symmetrical.*

SUM OF ANGLES OF A SPHERICAL TRIANGLE

342. Theorem XVI. *The sum of the angles of a spherical triangle is less than six right angles and greater than two right angles.*
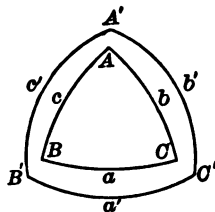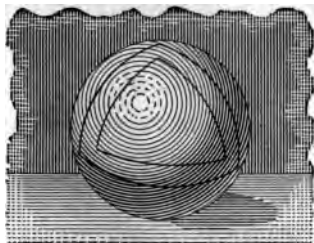
Given the spherical triangle *ABC*.

To prove that (1) $\angle A + \angle B + \angle C < 6$ *rt. angles.*

(2) $\angle A + \angle B + \angle C > 2$ *rt. angles.*

Proof: Construct the polar triangle $A'B'C'$, with sides a', b', c'.

(1) By § 339 $\angle A + \angle B + \angle C + a' + b' + c' = 6$ rt. △.

Since $a' + b' + c' > 0$, ∴ $\angle A + \angle B + \angle C < 6$ rt. △.

(2) Using § 322, show that $\angle A + \angle B + \angle C > 2$ rt. △.

343. Corollary 1. *State and prove the theorem on trihedral angles which corresponds to Theorem XVI.*

344. Corollary 2. *The sum of the angles of a spherical polygon of n sides is greater than* $2(n-2)$ *right angles and less than* $2n$ *right angles.*

Proof: Divide the polygon into $n - 2$ triangles. Then by § 342 the sum of the angles $> 2(n-2)$ rt. △.

Since each angle is less than two right angles, it follows that the sum is less than $2n$ right angles.

345. Corollary 3. *State and prove the theorem on polyhedral angles which corresponds to Corollary 2.*

EQUAL AND SYMMETRICAL SPHERICAL TRIANGLES

346. THEOREM XVII. *Two triangles on the same sphere, or on equal spheres, are equal or symmetrical if two angles and the included side of one are equal respectively to two angles and the included side of the other.*

Proof. By §§ 340 and 329 the polar triangles of the given triangles are equal or symmetrical. Hence, by § 341, the given triangles themselves are equal or symmetrical.

347. COROLLARY. *State and prove the theorem on trihedral angles which corresponds to Theorem XVII.*

348. THEOREM XVIII. *Two triangles on the same sphere, or on equal spheres, are equal or symmetrical if the angles of one are equal respectively to the angles of the other.*
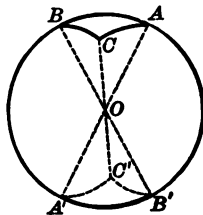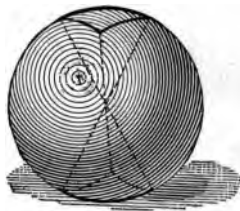
Proof: By §§ 340 and 328 the polar triangles of the given triangles are equal or symmetrical. Hence, by § 341, the given triangles themselves are equal or symmetrical.

349. COROLLARY. *State and prove the theorem on trihedral angles which corresponds to Theorem XVIII.*

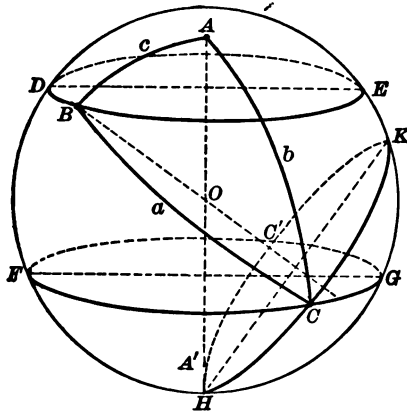Is there a theorem on plane triangles corresponding to that of § 348?

CONSTRUCTION OF SPHERICAL TRIANGLES

350. PROBLEM. *On a given sphere to construct a spherical triangle when its sides are given.*

 Solution. Let O be the given sphere, and a, b, c the sides of the required triangle, and let AA' be any diameter of the sphere. With A as a pole, construct circles DBE and FCG, whose polar distances from A are c and b respectively.

With B as a pole, construct a circle HCK, whose polar distance from B is a. Then construct the three great circle arcs, AB, BC, CA. ABC is the required triangle. Why?

SIGHT WORK

1. What restrictions if any is it necessary to impose upon the three given sides of the triangle in § 350? (See §§ 321, 322.)

2. In plane geometry two equal triangles may be constructed upon the same base and on the same side of it. Is a corresponding construction possible on the sphere?

3. If in the above construction each of two sides of the required triangle is very great, that is, nearly a semicircle, show from the construction that the third side must be very small.

CONSTRUCTION OF SPHERICAL TRIANGLES

351. PROBLEM. *To construct a spherical triangle when its three angles are given.*

Solution. Let the three given angles be A, B, C, and let a', b', c' be arcs such that $a' + \angle A = 180°$, $b' + \angle B = 180°$, and $c' + \angle C = 180°$. Then the triangle whose arcs are a', b', c' will be the polar triangle of the required triangle. This latter triangle $A'B'C'$ may be constructed by the method of § 350. Then construct the polar triangle of $A'B'C'$, which will be the required triangle.

Give reasons in full for each step.

352. PROBLEM. *To construct a trihedral angle when its face angles are given.*

Solution. Construct the corresponding spherical triangle by the method of § 350.

Give the construction in full and prove each step.

353. PROBLEM. *To construct a trihedral angle when its dihedral angles are given.*

Solution. Construct the corresponding spherical triangle by the method of § 351.

Give reasons in full for each step.

SIGHT WORK

1. If two spherical triangles having angles respectively equal are constructed on the same sphere, how are these triangles related ? Prove.

2. If two trihedral angles with face angles respectively equal are constructed as in § 352, how are they related ? Prove.

3. If two trihedral angles with dihedral angles respectively equal are constructed as in § 353, how are the trihedral angles related ? Prove.

4. What restrictions if any must be placed upon the given angles A, B, C in § 351? Compare Example 1, page 129.

5. What restrictions if any are needed in Examples 2 and 3 ?

EXERCISES

1. If two angles of a spherical tri-angle are unequal, the sides opposite them are unequal, the greater side being opposite the greater angle.
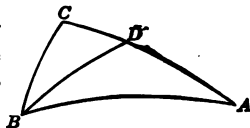
Suggestion. In the triangle ABC let $\angle B$ be greater than $\angle A$. Draw $\widehat{BD}$, making $\angle ABD = \angle A$.

Then, $\qquad \widehat{AD} = \widehat{BD}$, and $\widehat{BD} + \widehat{DC} > \widehat{BC}$.

Hence, show that $\qquad \widehat{AC} > \widehat{BC}$.

2. State and prove a theorem on trihedral angles corresponding to the preceding.

3. If the sides of a spherical triangle are 60°, 80°, 120°, find the angles of its polar triangle.

4. If the angles of a spherical triangle are 72°, 104°, 88°, find the sides of the polar triangle.

5. If a triangle is isosceles, prove that its polar triangle is isosceles.

6. If each side of a spherical triangle is a quadrant, describe its polar triangle.

7. Is it possible to construct a spherical triangle whose sides are 50°, 60°, 120°?

8. Is it possible to construct a spherical triangle whose sides are 100°, 120°, 150°?

Suggestion. Consider the polar triangle of such triangle.

9. Consider the questions on trihedral angles corresponding to the two preceding.

10. If the sides of a spherical triangle are 75°, 95°, and 115° respectively, find the angles of each triangle formed by the polar construction.

11. If it is given that a spherical triangle is equilateral, can we infer from the theorems thus far proved that its polar triangle is equilateral?
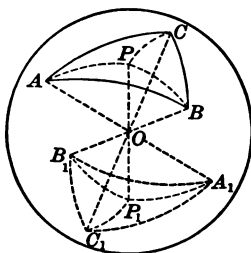
SYMMETRICAL TRIANGLES ARE EQUAL IN AREA

354. Equal Areas Defined. Two spherical polygons are said to *have equal areas* if they can be made to coincide, or if they can be divided into parts which can be made to coincide in pairs.

Compare this with the definition of equal areas in Plane Geometry.

355. THEOREM XIX. *Two symmetrical spherical triangles are equal in area.*

Proof: Let ABC be one of the given triangles. Extend the radii AO, BO, CO to meet the sphere in A_1, B_1, C_1, thus forming a triangle symmetrical to $\triangle ABC$. § 327

Let P be a pole of the circle through A, B, C. Extend PO to meet the sphere in P_1. Draw $\overarc{PA}$, $\overarc{PB}$, $\overarc{PC}$, and $\overarc{P_1A_1}$, $\overarc{P_1B_1}$, and $\overarc{P_1C_1}$.

Suppose that P lies within $\triangle ABC$.

The spherical triangles PAB, PBC, PCA, $P_1A_1B_1$, $P_1B_1C_1$, $P_1C_1A_1$ are all isosceles. § 296

Now prove

(1) $\triangle PAB = \triangle P_1A_1B_1$;   (2) $\triangle PBC = \triangle P_1B_1C_1$;

(3) $\triangle PCA = \triangle P_1C_1A_1$.

$\therefore$ area $\triangle ABC =$ area $\triangle A_1B_1C_1$.

But $\triangle A_1B_1C_1$ is equal to any other triangle which is symmetrical to $\triangle ABC$.

$\therefore$ Two symmetrical spherical triangles are equal in area.

BIRECTANGULAR SPHERICAL TRIANGLES

356. A **birectangular spherical triangle** is one having two right angles, as $\angle PAB$ in the figure.
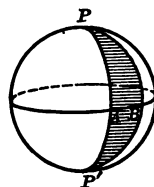
If the third angle of a birectangular triangle is 1°, the triangle contains one of 720 equal parts of the surface of the sphere.

357. Spherical Degree. The area of a birectangular triangle having a vertex angle of one degree is called a *spherical degree* and is used as a unit of measure of areas of spherical polygons.
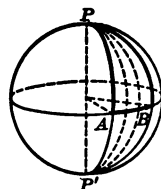
In a similar manner we define a *spherical minute* and a *spherical second.*

358. The Lune. A *lune* is a figure formed by two great semicircles having the same endpoints. The angle between these semicircles is the *angle of the lune.*

359. COROLLARY 1. *The area of a birectangular spherical triangle in terms of spherical degrees is equal to the number of degrees in the third angle of the triangle.*

360. COROLLARY 2. *The area of a lune in terms of spherical degrees is twice the number of degrees in the angle of the lune.*

361. Spherical Excess. The number of degrees by which the sum of the angles of a spherical triangle exceeds 180° is called the *spherical excess* of the triangle.

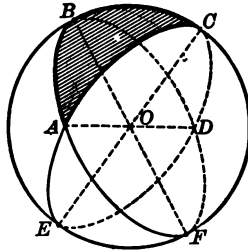The spherical excess of a spherical polygon is the sum of its angles less $(n-2)180°$, where n is the number of sides of the polygon.

EXERCISE

Show that the spherical excess of a spherical polygon is less than four right angles.

AREA OF A SPHERICAL TRIANGLE

362. THEOREM XX. *The area of a spherical triangle in terms of spherical degrees is equal to its spherical excess.* *

Proof: We are to show that area $\triangle ABC = \angle A + \angle B + \angle C - 180°$.

By § 360, the areas of the lunes $ACDB$, $CAEB$, $BCFA$ in spherical degrees are as follows:

$$\triangle ABC + \triangle BCD = ACDB = 2 \angle A.$$
$$\triangle ABC + \triangle BAE = CAEB = 2 \angle C.$$
$$\triangle ABC + \triangle CFA = BCFA = 2 \angle B.$$

Hence, adding,

$3 \triangle ABC + \triangle BCD, BAE, CFA = 2(\angle A + \angle B + \angle C)$.

Now $\triangle BCD$ and AEF are symmetrical and equal in area.

Hence,

$2 \triangle ABC + \triangle ABC, AEF, BAE, CFA = 2(\angle A + \angle B + \angle C)$.

But $\triangle ABC, AEF, BAE, CFA$ together constitute a hemisphere or 360 spherical degrees.

Hence, $2 \triangle ABC + 360° = 2 (\angle A + \angle B + \angle C)$.

Solving, $\quad \triangle ABC = \angle A + \angle B + \angle C - 180°$. Q. E. D.

NOTE. — The spherical degree differs fundamentally from the units of measure hitherto used. This unit is a certain fraction of the surface of the sphere and hence its actual size depends upon the size of the sphere.

* This theorem was discovered by Cavalieri. See *Frontispiece.*

AREA OF A SPHERICAL POLYGON

363. THEOREM XXI. *The area of a spherical polygon in terms of spherical degrees is equal to its spherical excess.*

Proof: Join one vertex of the polygon to each non-adjacent vertex, thus forming $n - 2$ spherical triangles. Now since the sum of the spherical excesses of these triangles is the spherical excess of the polygon, the conclusion is evident.

SIGHT WORK

1. What is the area in spherical degrees of a birectangular triangle if one of its angles is 54° ? if one angle is 24° ? if one angle is 36° ?

2. What is the spherical excess of a triangle the sum of whose angles is 285° ?

3. What is the spherical excess of a triangle whose angles are 75°, 110°, and 150° ?

4. What is the spherical excess of a spherical hexagon the sum of whose angles is 1060° ?

5. What is the spherical excess of a spherical pentagon whose angles are 90°, 120°, 75°, 108°, and 165° ?

6. What is the angle of a lune whose area is 160 spherical degrees.

7. The spherical excess of a triangle is 120°. Two of its angles are 110° and 108° respectively. Find the third angle.

8. Between what limits is the sum of the angles of a spherical polygon of eight sides ?

9. If the sum of the angles of a spherical polygon is 11 right angles, what is known about the number of its sides ?

10. If the sum of the angles of a spherical polygon is 14 right angles, what is known about the number of its sides ?

11. The sides of a spherical triangle are 85°, 95°, 110°. Find the area of each of the eight triangles formed by the polar construction from this triangle.

12. The area of a spherical triangle is 74 spherical degrees. One angle is 105°. Of the other two angles one is twice the other. Find all the angles of the triangle.

LATERAL AREA OF A FRUSTUM OF A CONE

364. THEOREM XXII.   *The lateral area of a frustum of a right circular cone is equal to the altitude of the frustum multiplied by the circumference of a circle whose radius is the perpendicular distance from a point in the axis of the frustum to the middle point of an element.*
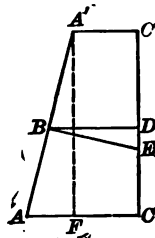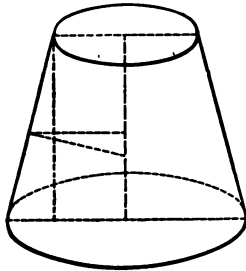
Given a frustum with an element AA' whose middle point is B, CC' the axis of the frustum, $BD \perp CC'$ and $EB \perp AA'$.

To prove that the lateral area is equal to $2 \pi \cdot EB \cdot CC'$.

Proof: By § 269, the lateral area is $2 \pi \cdot BD \cdot AA'$.

Hence, we must show that
$$EB \cdot CC' = BD \cdot AA'.$$

To do this, draw $A'F \perp AC$ and show that $\triangle AFA' \sim \triangle EDB$.

365. COROLLARY.   *The lateral area of a right circular cone is equal to its altitude times the length of a circle whose radius is the perpendicular from a point in the axis to the middle point of an element.*
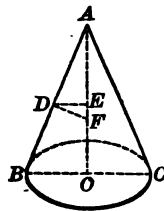
Note. The above theorem and corollary are needed in deducing the area of the surface of a sphere. We have already computed the area of a spherical triangle in terms of *spherical degrees*, but we now wish to derive the area of the spherical surface in terms of *plane units of measure.*

CIRCUMSCRIBED AND INSCRIBED CONES AND FRUSTUMS

**366.** About a circle circumscribe a polygon as follows : Construct two diameters *AB* and *CD* at right angles to each other and divide each quadrant into an *even* number of parts by points, as *E, F, G*. At each alternate division point, beginning with the first point *E*, draw a tangent.

There results a regular polygon with two vertices on each of the diameters *AB* and *CD* extended.

If now we construct another polygon in the same manner by dividing each quadrant into *twice* as many arcs, it will likewise have two vertices on each of the diameters *AB* and *CD* extended. The vertices on *AB* will lie between *M* and *N*.
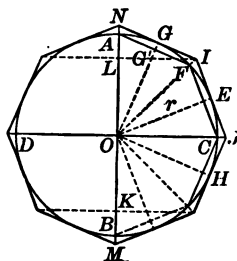
This construction may be repeated at pleasure, thus obtaining a sequence of polygons each lying closer to the circle than the preceding.

Now inscribe a polygon similar to the first one of the set just circumscribed, by joining the points *C* and *F, F* and *A*, and so on, repeating this process to form a sequence of inscribed polygons.

If now the whole figure is made to revolve about *AB* as an axis, the circle generates a sphere, and the circumscribed and inscribed polygons generate sets of circumscribed and inscribed cones and frustums of cones.

**367. Fundamental Assumption on the Area of a Sphere.** We assume that

A sphere has a definite area which is less than the surface of any circumscribed figure and greater than the surface of any inscribed convex figure.

The student should note that while the statement just preceding is obviously true, it is not capable of proof by pure deduction. It is therefore *necessarily* in the nature of an *assumption* or *axiom*.

368. THEOREM XXIII. *The area of a sphere whose radius is r is $4\,\pi r^2$.*

Proof: (a) Denote by r the radius of the circle which generates the sphere. That is, in the figure, $r = OG = OE = OH$.
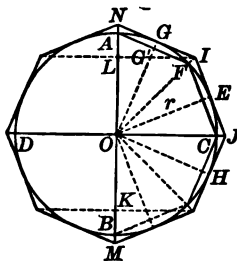
By § 364 the lateral areas of the outside frustums whose axes are OL and OK are $2\,\pi r \cdot OL$ and $2\,\pi r \cdot OK$, and by § 365 the lateral areas of the cones whose axes are LN and KM are respectively $2\,\pi r \cdot LN$ and $2\,\pi r \cdot KM$.

Hence, the total surface of the whole circumscribed figure is

$$2\,\pi r \cdot MK + 2\,\pi r \cdot KO + 2\,\pi r \cdot OL + 2\,\pi r \cdot LN,$$

or $\qquad 2\,\pi r(MK + KO + OL + LN) = 2\,\pi r \cdot MN.$

If now a polygon of twice the number of sides is constructed, as described in § 366, we obtain another circumscribed figure whose area is $2\,\pi r \cdot M'N'$ where M' and N' are the two vertices on the line AB extended, but lying between M and N, so that $2\,\pi r \cdot M'N' < 2\,\pi r \cdot MN$.

As this process goes on, the total surface generated decreases and may be made to approximate *as nearly as we please* to

$$2\,\pi r \times AB = 2\,\pi r \times 2\,r = 4\,\pi r^2.$$

Therefore, the area of the sphere cannot be greater than $4\,\pi r^2$.

(b) Using the same figure, let OG' be the apothem of the inscribed polygon.

Then, as under (a), we find that the area of the figure developed by revolving the inscribed polygon about AB is

$$2\,\pi \cdot AB \cdot OG' = 4\,\pi r \cdot OG'.$$

By continuing to double the number of sides, OG' may be made to approach as nearly as we please to $OG = r$, OG' being always less than OG.

Hence, the total surface developed increases and approaches *as nearly as we please* to

$$4\,\pi r \times OG = 4\,\pi r^2.$$

Therefore the area of the sphere cannot be less than $4\,\pi r^2$.

Since from (a) and (b) the area of the surface of the sphere can be neither greater than $4\,\pi r^2$ nor less than $4\,\pi r^2$, it follows that it is exactly equal to $4\,\pi r^2$.

Q. E. D.

SIGHT WORK

Use the value, $\pi = 3.1416$.

1. Find the surface of a sphere whose radius is one inch.

2. Find the surface of a sphere whose radius is 10 inches.

3. What is the relation between the surface of a sphere and the area of a circle of the same radius?

4. Find the area in square inches of a spherical degree on a sphere of radius 15 inches.

5. What fraction of the surface of a sphere is occupied by a birectangular triangle having one angle of 65°? by a lune with an angle of 84°?

6. Find the area in square inches of a birectangular spherical triangle with one angle equal to 35°, if the triangle is on a sphere of radius 10 inches.

7. Find the area in square inches of a lune whose angle is 42°, if the lune is on a sphere of radius 20 inches.

Example. Find the area in square inches of a spherical triangle whose angles are 80°, 85°, and 97° if the radius of the sphere is 6 inches.

Solution. The spherical excess of the triangle is

$$80° + 85° + 97° - 180° = 262° - 180° = 82°.$$

The total area of the sphere is $4\pi \times 6^2 = 452.3904$ sq. in., and one spherical degree is $\frac{1}{720}$ of $452.3904 = .62832$ sq. in.

Hence 82 spherical degrees is $82 \times .62832 = 51.52224$ sq. in.

1. Find the area in square inches of a spherical triangle whose angles are 70°, 80°, 90° if the radius of the sphere is 10 inches.

Suggestion. First find the spherical excess of the triangle.

2. Find the area in square inches of a spherical triangle whose angles are 95°, 110°, 75° if the radius of the sphere is 15 inches.

3. Find the area of a spherical polygon whose angles are 110°, 120°, 130°, and 95° if the radius of the sphere is 8 inches.

4. Find the area of a spherical polygon whose angles are 130°, 140°, 110°, 100°, 160°, and 150° if the radius of the sphere is 20 inches.

EXERCISES IN ALGEBRAIC COMPUTATION

1. The area of a sphere is s square inches. Find the radius of the sphere in terms of s.

2. The spherical excess of a spherical triangle is e and the radius of the sphere is r. Find the area of the triangle in terms of e and r.

3. The spherical excess of a spherical polygon is e and its area is a square inches. Find the radius of the sphere in terms of e and a.

4. The area in square inches of a spherical triangle is a and the radius of the sphere is r. Find the spherical excess in terms of a and r.

THE VOLUME OF A SPHERE BY INSPECTION

369. The formula for the volume of a sphere may be inferred directly from the accompanying figure.

The sphere is covered with a network of spherical quadrilaterals. If these are taken small enough, they may be regarded as approximately *plane surfaces.*

On this supposition we have a set of pyramids with a common altitude r and the sum of their bases approximately equal to the area of the sphere.
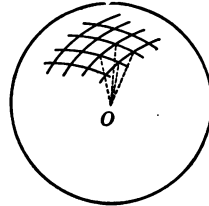
Hence, their combined volume is $\frac{1}{3} r \times$ (area of sphere) or $\frac{1}{3} r \cdot 4 \pi r^2$. That is, the volume is $\frac{4}{3} \pi r^3$.

It is clear that, by making these quadrilaterals sufficiently small each one may be made to approach as nearly as we please to a plane surface. Hence, each pyramidal figure with vertex at O is made to approach the form of a true pyramid as nearly as we please.

370. Fundamental Assumption on the Volume and Surface of a Sphere. In the formal proof on page 142, we use polyhedrons circumscribed about the sphere and we assume that

The surface and the volume of a sphere may be approximated as nearly as we please by taking the surfaces and the volumes of a series of circumscribed polyhedrons all of whose faces are made to decrease indefinitely.

Note. This assumption, when taken together with § 367, implies that the surface obtained by taking the surfaces of a series of polyhedrons is the same as the surface obtained by taking the surfaces of the circumscribed figures described in § 366. This is of course obvious at a glance, though a formal deductive proof is very difficult.
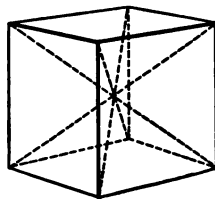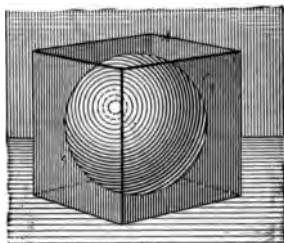
EXERCISE

Using § 369 find the volume of a sphere whose radius is 10 in.; also one whose diameter is 6 ft.

371. THEOREM XXIV. *The volume of a sphere whose radius is r is $\frac{4}{3}\pi r^3$.*

Proof: Consider a sphere of radius r with a cube circumscribed about it. Connect its vertices with the center of the

sphere. Then the cube is divided into six pyramids, each having an altitude r. Since the volume of each pyramid equals the area of its base times one third its altitude, it follows that the sum of the volumes of these pyramids is equal to the sum of their bases times one third their common altitude.

Consider now a polyhedron obtained from this cube by cutting off its eight vertices by planes tangent to the sphere. This new polyhedron is circumscribed about the sphere and approaches it more nearly than the cube. Continuing in this manner we may obtain a series of circumscribed polyhedrons approaching the sphere as nearly as we please.

By joining the vertices of each polyhedron to the center of the sphere we obtain a set of pyramids, each with altitude r.

Suppose the total surfaces of these successive polyhedrons are s_1, s_2, s_3, s_4, $\cdots$. Then their volumes are $\frac{r}{3}s_1$, $\frac{r}{3}s_2$, $\frac{r}{3}s_3$ $\cdots$ But, by § 370, s_1, s_2, s_3, $\cdots$ approach the surface of the sphere, or $4\pi r^2$, as nearly as we please. Hence the successive volumes approach $\frac{r}{3} \times 4\pi r^2 = \frac{4}{3}\pi r^3$ as nearly as we please. That is,

Surface of a sphere $= 4\pi r^2$, and Volume of a sphere $= \frac{4}{3}\pi r^3$.

SIGHT WORK

1. Given a sphere of radius 6 inches, is there any *upper* limit to the volume of its circumscribed polyhedrons ? That is, can polyhedrons be circumscribed having a volume as large as we please ?

2. With the same sphere is there any *lower* limit to the volume of its circumscribed polyhedrons ?

3. Show that the areas of two spheres are in the same ratio as the squares of their radii or of their diameters.

4. Show that the volumes of two spheres are in the same ratio as the cubes of their radii or of their diameters.

5. Express the area of a sphere whose radius is 8 in. in terms of π.

6. Express the volume of a sphere whose radius is 10 ft. in terms of π.

7. The surface of a polyhedron circumscribed about a sphere of radius 4 in. is 420 sq. in. Find its volume.

EXERCISES IN ARITHMETIC COMPUTATION

1. The volume of a polyhedron circumscribed about a sphere of radius 3.5 in. is 450 cu. in. Find its surface.

2. If the area of a sphere is 227 sq. ft., find its radius.

3. If the volume of a sphere is 335 cu. in., find its radius.

4. If the volumes of two spheres are 27 cu. in. and 729 cu. in., find the ratio of their radii.

EXERCISES IN ALGEBRAIC COMPUTATION

1. The volume of a sphere is v cubic inches. Find its radius in terms of v and π.

2. The area of the surface of a sphere is s square inches. Find its volume in terms of s and π.

3. The area of the surface of a sphere is numerically equal to its volume. Find the radius of the sphere if an inch is the unit of measure.

4. The difference between the volume of a cube and that of its inscribed sphere is v. Find the radius of the sphere in terms of v and π.

SPHERICAL SEGMENTS, CONES, SECTORS

372. Zone, Segment. That part of a spherical surface included between two parallel planes cutting it is called a *zone*. The perpendicular distance between the planes is the *altitude* of the zone and of the corresponding segment.

The portion of a sphere included between two parallel planes cutting it is called a *spherical segment*, and the two circular sections made by the parallel planes are its *bases*.

If one of the cutting planes is tangent to the sphere, then the spherical segment and the corresponding zone are said to have but *one base*. The altitude in this case is the perpendicular distance from the base to the tangent plane.

373. Spherical Cone, Sector. If one nappe of a convex conical surface has its vertex at the center of a sphere, the portion of the sphere cut out by this surface is called a *spherical cone*.
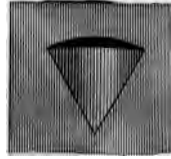
If two spherical cones have the same axis, one lying within the other, the figure formed by their two lateral surfaces, together with the part of the sphere intercepted between them, is called a *spherical sector*.


If the two cones are right circular cones, they intercept circles on the sphere, and the zone thus included is called the *base* of the spherical sector.

If the accompanying figure be revolved about *LM* as an axis, then any arc, as *GD* or *MD*, generates a zone, the former with two bases, the latter with one.

The figure *MDF* or *FDGE* generates a spherical segment, the former with one base, the latter with two.

The figure *CAL* or *CBL* generates a spherical cone, *CL* being the common axis.

The figure *CBA* generates a spherical sector and arc *AB* generates the zone which is the base of the spherical sector.
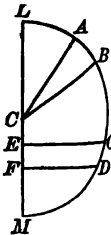
AREA OF A ZONE. VOLUME OF A SPHERICAL CONE

374. Area of a Zone. An argument precisely like that of § 368 shows that the area of a zone is

$$s = 2\,\pi r h,$$

where *h* is the altitude of the zone.

That is, instead of *AB*, the diameter in case of the sphere, we should have the sum of the altitudes of the frustums circumscribed about the zone equal to *h*, the altitude of the zone.

375. Volume of a Spherical Cone and a Spherical Sector. An argument precisely like that of § 371 shows that the volume of a spherical cone is

$$v = \frac{r}{3} \cdot s,$$

where *s* is the area of the zone of one base which is cut out of the sphere by the cone. Hence, if *h* is the altitude of this zone, we have

$$v = \frac{r}{3} \cdot 2\,\pi r h = \frac{2\,\pi}{3}\,r^2 h.$$

In like manner the *volume of a spherical sector* is

$$v = \frac{2\,\pi}{3}\,r^2 h,$$

where *h* is the altitude of the zone of two bases which is cut out by the sector.

EXERCISES

1. The radius of a sphere is 6 in. and the altitude of a zone is 5 in. Find the area of the sphere and of the zone.

2. The area of a zone is $36\,\pi$ sq. ft. and its altitude 4 ft. Find the radius of the sphere.

3. On a sphere of radius 8 in. a spherical cone cuts out a zone whose altitude is 2 in. Find the volume of the cone.

4. Find the volume of the spherical sector cut out of a sphere of radius 9 in., if the altitude of the zone is 2 in.

VOLUME OF A SPHERICAL SEGMENT

376. Problem. *To find the volume of a spherical segment.*

Solution. Let r be the radius of the sphere, and r_1 and r_2 the radii of the bases of the segment, h the altitude of the segment, and let the segment be generated by revolving the figure $ACDB$ about AO as an axis.
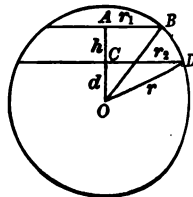
We have Vol. generated by $ODB = \dfrac{2\pi}{3} r^2 h.$ § 375

Vol. generated by $OAB = \dfrac{\pi}{3} r_1^2 (h + d).$ (Why?)

Vol. generated by $OCD = \dfrac{\pi}{3} r_2^2 d.$ (Why?)

Hence, $v = \dfrac{2\pi}{3} r^2 h + \dfrac{\pi}{3} r_1^2 (h + d) - \dfrac{\pi}{3} r_2^2 d$

$$= \dfrac{\pi}{3} [2\, r^2 h + r_1^2 h + d(r_1^2 - r_2^2)]. \qquad (1)$$

From $r^2 = r_2^2 + d^2$ and $r^2 = r_1^2 + (h + d)^2$

we obtain $\qquad d = \dfrac{r_2^2 - r_1^2 - h^2}{2\,h}.$ (2)

Substituting this value of d in $r^2 = r_2^2 + d^2,$

we get $\qquad r^2 = \dfrac{r_2^4 + r_1^4 + h^4 - 2\, r_1^2 r_2^2 + 2\, h^2 r_2^2 + 2\, h^2 r_1^2}{4\,h^2}.$ (3)

Substituting (2) and (3) in (1) and reducing, we have

$$v = \dfrac{\pi h}{2}(r_1^2 + r_2^2) + \dfrac{\pi}{6} h^3. \qquad \text{Q. E. F.}$$

377. Corollary. *The volume of a spherical segment of one base is*

$$v = \pi h^2 \left(r - \dfrac{h}{3} \right).$$

Suggestion. In this case $r_1^2 = 0$ and $r_2^2 = r^2 - (r - h)^2 = 2\,rh - h^2.$ Substituting in the formula above, we have the result.

<div align="center">SUMMARY OF BOOK V</div>

1. Define sphere, diameter, radius.

2. Collect the theorems of Book V involving plane sections of the sphere.

3. Define axis and pole of a circle, great circle, polar distance.

4. Define tangent plane to a sphere, inscribed and circumscribed polyhedrons.

5. Arrange in parallel columns the corresponding theorems on trihedral angles and spherical triangles which are proved without the use of polar triangles.

6. Collect the definitions on polar triangles.

7. Collect the theorems on polar triangles.

8. Continue the lists begun in Example 5, adding the theorems proved by means of polar triangles.

9. Make a list of the definitions involving polyhedral angles and spherical polygons.

10. Collect the theorems involving polyhedral angles and spherical polygons.

11. Collect the theorems on the areas of spherical triangles and polygons.

12. Give the definitions and assumptions pertaining to the area and volume of the sphere.

13. State all the theorems pertaining to the area and volume of the sphere.

14. Give the definitions and theorems pertaining to spherical figures, such as zones, cones, sectors, segments.

15. Collect all the mensuration formulas in this Book.

16. Collect all the mensuration formulas of solid geometry.

17. Describe some of the most important applications in this Book. Return to this question after studying the following sets of problems.

PROBLEMS ON BOOK V

1. What part of the earth's surface lies in the torrid zone? What part in the temperate zones? What part in the frigid zones? The parallels $23\frac{1}{2}°$ north and south of the equator are the boundaries of the torrid zone, and the parallels $66\frac{1}{2}°$ north and south are the boundaries of the frigid zones.

**2.** Find to four places of decimals the area of a sphere circumscribed about a cube whose edge is 6. No square root is to be approximated in the process, and the value of π is taken as 3.1416.

3. Can the volume of the sphere in the preceding exercise be approximated without finding a square root? Find the volume.

4. Find the area of a sphere circumscribed about a rectangular parallelopiped whose sides are a, b, and c.

5. Find the volume of the sphere in the preceding example.

6. A fixed sphere with center O has its center on another sphere with center O'. Show that the area of the part of O' which lies within O is equal to the area of a great circle of the sphere O, provided the radius of the sphere O is not greater than the diameter of O'.
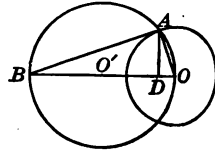
Suggestion. Let the figure represent a cross section through the centers of the two spheres. Connect O with A and B. Then $\overline{OA}^2 = OB \times OD$. But OD is the altitude of the zone of O' which lies within O, and OB is the diameter of the sphere O'. Hence, the area of the zone is $\pi OB \times OD = \pi \overline{OA}^2$.

7. Given a solid sphere of radius 12 inches. A cylindrical hole is bored through it so that the axis of the cylinder passes through the center of the sphere. What area of the sphere is removed if the diameter of the hole is 4 inches?

8. Find the volume removed from the sphere by the process described in the preceding exercise.

9. A cylindrical post 6 in. in diameter is surmounted by a part of a sphere 10 in. in diameter, as shown in the figure. Find the surface and the volume of the part of the sphere used.

10. A cylindrical post 5 ft. long and 4 in. in diameter is surmounted by a part of a sphere 9 in. in diameter as shown in the figure. Find the volume of the whole post including the spherical part.

11. Find the volume of a spherical shell one inch thick if its outer diameter is 8 inches.

12. Compare the volumes and areas of a sphere and the circumscribed cylinder.

13. In a sphere of radius r a cylinder is inscribed whose altitude is equal to its diameter. Compare its volume and area with those of the sphere.

14. Find the diagonal of a cube in terms of its side, and also a side in terms of half the diagonal.

15. Express the volume of a cube inscribed in a sphere in terms of the radius of the sphere.

16. Three spheres each of radius r are placed on a plane so that each is tangent to the other two. A fourth sphere of radius r is placed on top of them. Find the distance from the plane to the top of the upper sphere.

17. Find the vertical distance from the floor to the top of a triangular pile of spherical cannon balls, each of radius 5 inches, if there are 3 layers in the pile.

18. Solve a problem like the preceding if there are 16 layers in the pile, each shot of radius r.

FORMULAS DEVELOPED IN BOOKS III, IV, V

1. If V is the volume of a rectangular parallelopiped whose dimensions are a, b, c, then

$$V = abc.$$

2. If V is the volume, b the area of the base, and h the altitude of a prism or cylinder, then

$$V = hb.$$

3. If S is the lateral surface, p the perimeter of a right section of a prism or cylinder, and e the lateral edge or element, then

$$S = pe.$$

4. If V is the volume, b the area of base, and h the altitude of a pyramid or cone, then

$$V = \tfrac{1}{3} bh.$$

5. If S is the lateral area, p the perimeter of the base, and l the slant height of a regular pyramid or cone, then

$$S = \tfrac{1}{2} pl.$$

6. If V is the volume, b the lower base, b' the upper base, and h the altitude of a frustum of pyramid or cone, then

$$V = \tfrac{1}{3} h(b + b' + \sqrt{bb'}).$$

7. If S is the area of the surface, V the volume, and r the radius of a sphere, then

$$S = 4 \pi r^2 \text{ and } V = \tfrac{4}{3} \pi r^3.$$

8. If a is the spherical excess in degrees of a spherical polygon, S the area in square units, and r the radius of a sphere, then

$$S = \frac{a}{720} \cdot \frac{4}{3} r \pi^3.$$

9. If S is the area, h the altitude, and r the radius of a lune, then

$$S = 2 \pi r h.$$

10. If V is the volume of spherical cone, h the altitude, and r the radius of the sphere, then

$$V = \frac{2\pi}{3} r^2 h.$$

11. If V is the volume of a spherical segment, r the radius of the sphere, r_1 and r_2 the radii of the bases, and h the altitude of the segment, then

$$V = \frac{\pi h}{2}\left(r_1^2 + r_2^2\right) + \frac{\pi}{6} h^3.$$

SUPPLEMENTARY EXERCISES IN COMPUTATION

1. Find the volume of a rectangular parallelopiped if

(1) $a = 6$	(2) $a = \frac{7}{8}$	(3) $a = \sqrt{3}$	(4) $a = 3\sqrt{5}$
$b = 8$	$b = \frac{9}{10}$	$b = \sqrt{6}$	$b = \sqrt{15}$
$c = 7\frac{3}{4}$	$c = 3$	$c = 4$	$c = 3$

2. Find volume of a prism, or cylinder, if

(1) $b = 12$	(2) $b = \sqrt{3}$	(3) $b = 4\sqrt{2}$	(4) $b = 8\frac{1}{4}$
$h = 16$	$h = \sqrt{6}$	$h = \sqrt{10}$	$h = 4\sqrt{3}$

3. Find volume of a pyramid or cone if

(1) $b = 46$	(2) $b = 47\frac{1}{4}$	(3) $b = 34\pi$	(4) $b = 42\pi$
$h = 12$	$h = 10$	$h = 8$	$h = 6\sqrt{2}$

4. Find the lateral surface of a regular pyramid or cone if

(1) $p = 8$	(2) $p = 3\sqrt{7}$	(3) $p = \sqrt{7}$	(4) $p = 3\sqrt{2}$
$l = 7$	$l = \sqrt{21}$	$l = \sqrt{14}$	$l = 4\sqrt{6}$

5. Find the volume of a frustum of a pyramid or cone if

(1) $b = 8$	(2) $b = 2\sqrt{2}$	(3) $b = 16\pi$	(4) $b = 3\sqrt{2}$
$b' = 6$	$b' = \sqrt{3}$	$b' = 9\pi$	$b' = 6\sqrt{3}$
$h = 4$	$h = 6$	$h = 3$	$h = 8$

6. Find the volume or surface of a sphere if

$$r = 4, \ r = 8, \ r = 3\sqrt{2}, \ r = 7\sqrt{3}, \ r = 12\sqrt{6}$$

7. Find the area of a spherical polygon if

(1) $a = 84°$	(2) $a = 112° \ 30'$	(3) $a = 49° \ 25' \ 17''$
$r = 12$	$r = 8$	$r = 14$

MISCELLANEOUS REVIEW EXERCISES

A. LOCUS PROBLEMS

1. Find the locus of all points in space equally distant from two parallel lines and also from two parallel planes.   Discuss.

2. Find the locus of all points in space equally distant from each of two intersecting straight lines and also from two intersecting planes.   Discuss.

3. What is the locus of all points at a perpendicular distance of 2 feet from a given line and lying in a plane parallel to the line?   Discuss.

4. Find the locus of all points equidistant from two given points *A* and *B*, and also equidistant from two planes *M* and *N*.   Discuss.

5. What is the locus of all points on the floor of a room which are equally distant from two diagonally opposite corners of the room, one in the floor and one in the ceiling?

6. Find the locus of all points on a sphere where it is met by line-segments of equal length drawn from a fixed point *P* outside the sphere.   Discuss.

7. Find the locus of all points which are at the same fixed distance from each of two intersecting planes, *M* and *N*, and also equally distant from two planes, *P* and *Q*.

8. Given a plane *M* and a point *P* not in *M*.   Find the locus of a point which divides in a given ratio each segment connecting *P* with a point in *M*: (*a*) if the segments are divided internally; (*b*) if they are divided externally.

9. A segment *AB* of fixed length is free to move so that its end-points lie in two fixed parallel planes.   Find the locus of a point *C* on *AB* if *AC* is of fixed length.

10. Given two fixed points in space, through each of which passes a system of straight lines.   If each line of one system is perpendicular to a line of the other system, find the locus of the intersection points.

B. Problems in Numerical Computation

1. A pedestal for a monument is in the shape of a frustum of ʾa regular hexagonal pyramid, the radius of the upper base being 4 ft., that of the lower base 6 ft., and the altitude of the frustum 8 ft. Find its volume, slant height, and lateral surface.

2. The area of the lower base of a frustum of a pyramid is 42 sq. ft., its altitude 8 ft., and volume 200 cu. ft. Find the area of the upper base.

3. The area of the base of a pyramid is 480 sq. ft. and its altitude 30 ft. Find the volume of the frustum remaining after a pyramid with altitude 10 ft. has been cut off by a plane parallel to the base.

4. The area of the base of a pyramid is 250 sq. in. If a plane section of the pyramid parallel to the base and at a distance of 5 in. from it has an area of 175 sq. in., find the altitude of the pyramid.

5. The figure below represents a solid whose base is a rectangle 50 ft. long and 40 ft. wide. Its height is 12 ft. and its top a rectangle 20 ft. by 10 ft. Find its volume.
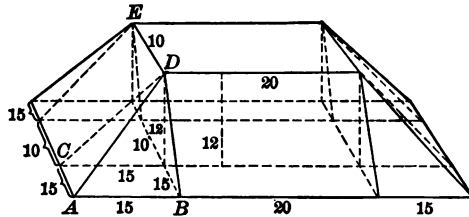
6. A frustum of a right circular cone has an altitude one half that of the cone. If its slant height is 8 ft. and lateral area 64 π sq. ft., find the diameters of its bases.

7. If the area of the base of a cone is 16 π sq. in. and its altitude 6 in., find the distance from the vertex to a plane, parallel to the base, which cuts out a section of area 9 π sq. in.

C. PROBLEMS IN ALGEBRAIC COMPUTATION

1. Find the total area and the volume of a regular tetrahedron each of whose edges is e.

2. If the numerical values of the volume and of the total area of a regular tetrahedron are equal, what is the length of its edge?

3. Find the length of an edge of a regular tetrahedron if its volume is numerically equal to the square of the edge.

4. Cut a pyramid of altitude h by means of a plane parallel to the base so that the perimeter of the section shall be one third that of the base.

5. If the altitude of a pyramid is h, how far from the base must a plane parallel to it be drawn so that the area of its cross section shall be half that of the base of the pyramid?

6. In a regular right pyramid a plane parallel to the base cuts it so as to make a section whose area is one half that of the base. Find the ratio between the lateral area of the pyramid and that of the small pyramid cut off by the plane.

7. If the diameter of a right circular cylinder is equal to its altitude, determine the diameter so that the total area of the cylinder shall be equal numerically to its volume.

8. Cut a right circular cone of altitude h by a plane parallel to the base so that the area of the section shall be one third that of the base. Find the distance from the vertex to the plane.

9. Show that the lateral area of the small cone cut off in Example 8 is one third the lateral area of the original cone.

10. In a right circular cone, with altitude h, and r the radius of its base, a cylinder is inscribed as shown in the figure. Find the radius OF of the cylinder if the area of the ring bounded by the circles OF and OA is equal to the lateral area of the small cone cut off at the top.
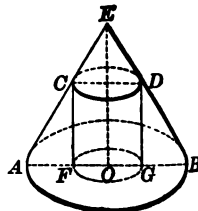
D. PROBLEMS IN CONSTRUCTION

1. Construct a plane tangent to a given sphere and parallel to a given plane. How many such planes are there?

Suggestion. From the center of the sphere pass a line perpendicular to the given plane and draw tangent planes at the points where this line meets the surface of the sphere. See §§ 306, 129.

2. Construct a plane tangent to a given sphere and perpendicular to a given line. How many such planes are there?

3. How many planes may be tangent to a sphere at a point on the sphere? How many lines? Show how to construct them.

4. Through a given point exterior to a sphere construct a line tangent to the sphere.

5. How many lines tangent to a sphere can be constructed from a point outside the sphere?

6. Through a given point exterior to a sphere construct a plane tangent to the sphere. How many planes can be passed through a given exterior point tangent to the sphere?

7. How many planes tangent to a sphere can be passed through two given points *A* and *B* outside a sphere? Discuss fully if the line *AB* (1) meets the sphere in two points; (2) is tangent to the sphere; (3) does not meet the sphere. Show how to make the constructions.

8. Is it possible to construct a spherical triangle each of whose angles is a right angle? Show how to construct one. In a trirectangular spherical triangle what is the length of each side in terms of degrees?

9. Construct a spherical triangle such that its polar triangle is identical with the given triangle.

10. Construct a spherical triangle whose sides are 70°, 80°, and 110°, respectively, and find the sides of each of the eight triangles formed by its polar construction.

E. Theorems to be Proved

1. Prove that planes perpendicular to the faces of a trihedral angle and bisecting its face angles meet in a line.

2. Prove that if in two tetrahedrons three faces of one are equal respectively to three faces of the other and similarly placed about a vertex, the tetrahedrons are equal.

3. Prove that two tetrahedrons are equal if two faces and the included dihedral angle are equal and similarly placed.

4. In any regular tetrahedron, an altitude equals three times the perpendicular from its foot to any face; or an altitude equals the sum of the perpendiculars to the faces from any point within the tetrahedron.

5. In a cylinder of revolution the diameter of whose base equals the altitude, the volume equals one third the product of the total surface by the radius of the base.

6. If two intersecting planes are each tangent to a cylinder, show that their line of intersection is parallel to an element of the cylinder and also parallel to the plane containing the two elements of contact.

7. The volume of a frustum of a right circular cone equals the sum of a cylinder and a cone of the same altitude as the frustum, and with radii which are respectively the half sum and the half difference of the radii of the frustum.

8. Of circles on a sphere whose planes pass through a given point within the sphere, the smallest is that circle whose plane is perpendicular to the diameter through the given point.

9. A right triangular prism is cut by a plane not parallel to the base, but such that its intersection DE is parallel to the base segment AB. Show that the volume of the part thus cut off is one third the product of the sum of the three vertical edges and the area of the base.
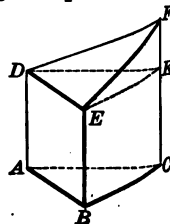
Adrien Marie Legendre (1752–1833) was a celebrated French mathematician. He was one of three commissioners who introduced the metric system in France, having also been a member of the commission for determining the length of the meter.

Besides many treatises on advanced subjects, he wrote a book on elementary geometry which was the most successful of the many attempts to supersede Euclid as a text-book. It went through many editions in French and was translated into almost every other civilized language.

APPENDIX

378. Similar Cylinders. Two right circular cylinders are *similar* if they are generated by similar rectangles revolving about corresponding sides.

379. THEOREM I. *If in two similar cylinders s and s' are the lateral areas, S and S' the total areas, V and V' the volumes, h and h' the altitudes, and r and r' the radii, then*

$$\frac{s}{s'}=\frac{S}{S'}=\frac{r^2}{r'^2}=\frac{h^2}{h'^2} \text{ and } \frac{V}{V'}=\frac{r^3}{r'^3}=\frac{h^3}{h'^3}.$$
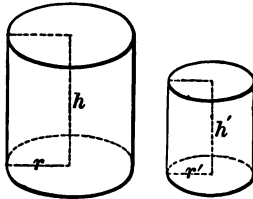
Suggestions for Proof: To find the ratios $\frac{s}{s'}$ and $\frac{S}{S'}$, make use of the following, giving reasons for each in detail.

(1) $s = 2\pi rh$, (2) $S = 2\pi r(r+h)$,

(3) $\frac{r}{r'}=\frac{h}{h'}$, and (4) $\frac{r+h}{r'+h'}=\frac{r}{r'}=\frac{h}{h'}$.

To find the ratio $\frac{V}{V'}$ make use of $V=\pi r^2h$, and $V'=\pi r'^2h'$, and $\frac{r}{r'}=\frac{h}{h'}$.

RATIOS RELATING TO SIMILAR CONES

380. Similar Cones. Two right circular cones are *similar* if they are generated by two similar right triangles revolving about corresponding sides.

381. THEOREM II. *If in two similar cones s and s' are the lateral areas, S and S' the total areas, V and V' the volumes, h and h' the altitudes, and r and r' the radii, then*

$$\frac{s}{s'} = \frac{S}{S'} = \frac{r^2}{r'^2} = \frac{h^2}{h'^2} \quad and \quad \frac{V}{V'} = \frac{r^3}{r'^3} = \frac{h^3}{h'^3}.$$
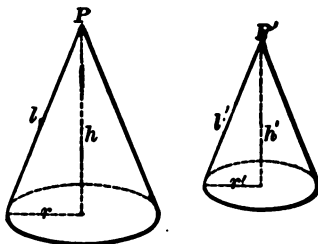
Suggestions for Proof: See suggestions under § 379.

EXERCISES

1. The lateral area of a cone is 36 square inches.   What is the lateral area of a similar cone whose altitude is ¾ that of the given cone?

2. The total area of one of two similar cones is three times that of the other.   Compare their altitudes and also their radii.

3. The sum of the total areas of two similar cones is 144 square inches.   Find the area of each cone if one is 1¼ times as high as the other.

4. The volume of one of two similar cones is 5 times that of the other.   Compare their altitudes.

CONDITIONS FOR SIMILARITY OF TETRAHEDRONS

382. Similar Polyhedrons. Two polyhedrons are *similar* if they have the same number of faces similar each to each and similarly placed, and have their corresponding polyhedral angles equal.

Any two parts which are similarly placed are called *corresponding parts*, as corresponding faces, edges, vertices.

383. THEOREM III. *Two tetrahedrons are similar if three faces of one are similar respectively to three faces of the other, and are similarly placed.*
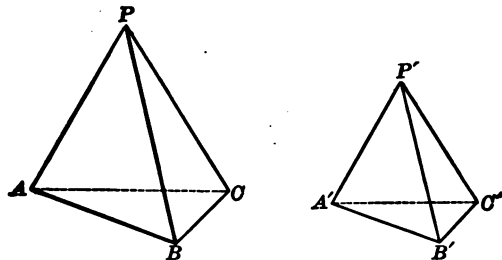
Given the tetrahedrons *P-ABC* and *P'-A'B'C'* having
$\triangle APB \sim \triangle A'P'B'$, $\triangle APC \sim \triangle A'P'C'$, and $\triangle BPC \sim \triangle B'P'C'$.

To prove $P\text{-}ABC \sim P'\text{-}A'B'C'$.

Proof: (1) Show that $\triangle ABC \sim \triangle A'B'C'$.

(2) Show that trihedral $\measuredangle P$ and P' are equal.

Likewise $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$.

Hence, by definition, the polyhedrons are similar.

SIGHT WORK

1. If the two prisms in the figure are similar, name the pairs of corresponding parts. Likewise for two similar pyramids.
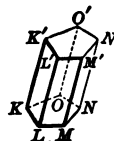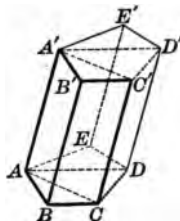
TETRAHEDRONS HAVING TWO EQUAL TRIHEDRAL ANGLES

384. THEOREM IV. *The volumes of two tetrahedrons, having a trihedral angle of the one equal to a trihedral angle of the other, are proportional to the products of the edges which meet in the vertices of these angles.*
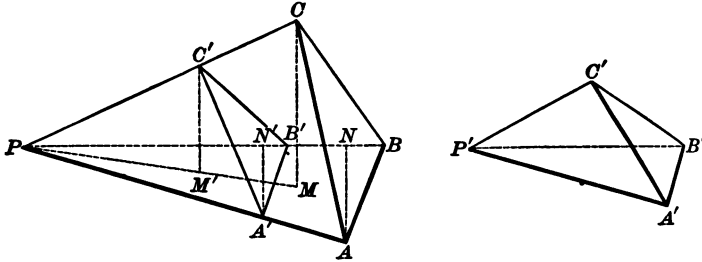
Given the tetrahedrons *P-ABC* and *P'-A'B'C'* whose volumes are *V* and *V'* and in which Tri. $\angle P$ = Tri. $\angle P'$.

To prove that $\dfrac{V}{V'} = \dfrac{PA \cdot PB \cdot PC}{P'A' \cdot P'B' \cdot P'C'}.$

Proof: Place *P'-A'B'C'* so that Tri. $\angle P'$ coincides with Tri. $\angle P$.

Let *CM* and *C'M'* be the altitudes of *P-ABC* and *P-A'B'C'* from the vertices *C* and *C'* upon the plane *PAB*.

Let *AN* and *A'N'* be the altitudes of the $\triangle$ *PAB* and *PA'B'*.

Then $\dfrac{V}{V'} = \dfrac{\frac{1}{3} CM \cdot \text{area } PAB}{\frac{1}{3} C'M' \cdot \text{area } PA'B'} = \dfrac{CM \cdot PB \cdot AN}{C'M' \cdot PB' \cdot A'N'}.$

Now prove $\dfrac{CM}{C'M'} = \dfrac{PC}{PC'}$ and $\dfrac{AN}{A'N'} = \dfrac{PA}{PA'}.$

Hence, we have $\dfrac{V}{V'} = \dfrac{PC \cdot PB \cdot PA}{P'C' \cdot P'B' \cdot P'A'}.$

Give all the steps and reasons in detail.

385. THEOREM V. *The volumes of two similar tetrahedrons are in the same ratio as the cubes of their corresponding edges.*

Given $P\text{-}ABC \sim P'\text{-}A'B'C'$, with volumes V and V'.

To prove that $\dfrac{V}{V'} = \dfrac{\overline{PA}^3}{\overline{P'A'}^3}$.

Proof: We have $\dfrac{V}{V'} = \dfrac{PA \cdot PB \cdot PC}{P'A' \cdot P'B' \cdot P'C'}$. § 384

Now use the properties of similar triangles to complete the proof. Use the figure of § 384.

EXERCISES

1. Show that two tetrahedrons are similar if they have a dihedral angle of one equal to a dihedral angle of the other and the including faces similar each to each and similarly placed.

2. Show that the total areas of two similar tetrahedrons are in the same ratio as the squares of any two corresponding edges.

3. Show that if each of two polyhedrons is similar to a third they are similar to each other.

4. Two similar tetrahedral mounds have a pair of corresponding dimensions 3 ft. and 4 ft. If one mound contains 40 cu. ft. of earth, how much does the other contain?

5. The edges of a tetrahedron are 3, 4, 5, 6, 7, and 8. Find the edges of a similar tetrahedron containing 64 times the volume.

6. Find what fraction of the altitude of a tetrahedron must be cut off by a plane parallel to the base, measuring from the vertex, in order that the new pyramid thus detached may have one fifth of the original volume.

FIGURES HAVING A CENTER OF SIMILITUDE

386. Center of Similitude. Two figures are said to have a *center of similitude* O, if for any two points, A and B, of the one figure the lines AO and BO meet the other in two points, A' and B', called *corresponding points*, such that

$$OA : OA' = OB : OB'.$$

See figures under §§ 387–392.

387. THEOREM VI. *Any two figures which have a center of similitude are similar.*

Proof: (1) *Two triangles.*

Given
$$\frac{OA}{OA'} = \frac{OB}{OB'} = \frac{OC}{OC'}.$$
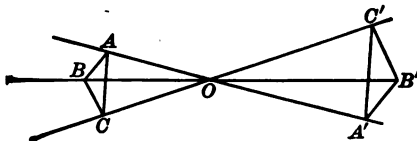
Let the student prove that $\triangle ABC \sim \triangle A'B'C'$.

In case the triangles do not lie in the same plane, use § 101 to show that the corresponding ∠s are equal.

(2) *Two polygons.*

Given
$$\frac{OA}{OA'} = \frac{OB}{OB'} = \frac{OC}{OC'}, \text{ etc.}$$
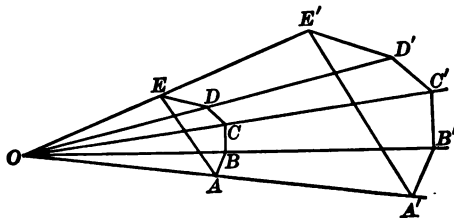
Give the proof both for polygons in the same plane and not in the same plane.

(3) *Two tetrahedrons.*
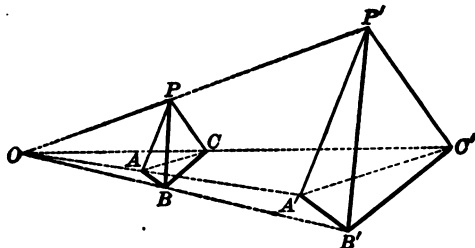
With the same hypothesis as before, we must prove $\triangle PAB \sim P'A'B'$, $\triangle PBC \sim \triangle P'B'C'$, $\triangle PCA \sim \triangle P'C'A'$, and then use § 383.

(4) *Any two polyhedrons.*

(*a*) Prove corresponding polygonal faces similar.

(*b*) Prove corresponding polyhedral angles equal.

The last step requires not only equal *face angles* about the vertex, as in the case of the tetrahedron, but also equal *dihedral* angles. Note that two dihedral angles are equal if their faces are parallel right face to right face and left face to left face. (Why?)

(5) Consider *any two figures whatsoever* having a center of similitude.

(*a*) Take any three points *A*, *B*, *C*, in one figure and the three corresponding points, *A'*, *B'*, *C'*, in the other.

Then *AB* and *A'B'*, *AC* and *A'C'*, etc., are called *corresponding linear dimensions*, and the triangles *ABC* and *A'B'C'* are *corresponding triangles*.

(*b*) It is clear that any two corresponding linear dimensions have the same ratio as any other two, and that any two corresponding triangles are similar.

In this sense any two figures having a center of similitude are thus proved to be *similar*.

SIMILAR TRIANGLES PLACED IN SIMILITUDE

388. Ratio of Similitude. The *ratio of similitude* of two similar figures is the common ratio of their corresponding linear dimensions. This ratio is the same as the *distance ratio* of corresponding points from the center of similitude.

389. THEOREM VII. *Two similar triangles may be so placed as to have a center of similitude.*
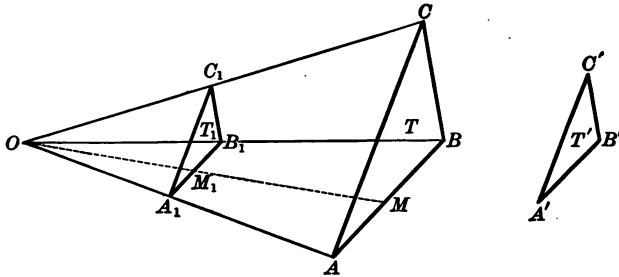
Given the similar triangles T **and** T', **in which**

$$\frac{A'B'}{AB} = \frac{A'C'}{AC} = \frac{B'C'}{BC}.$$

To prove that they may be placed with a center of similitude.

Proof: From any point O draw OA, OB, OC.
On these rays take A_1, B_1, C_1 so that

$$\frac{OA_1}{OA} = \frac{OB_1}{OB} = \frac{OC_1}{OC} = \frac{A'B'}{AB}.$$

Now show the following:

(1) $\triangle T_1 \sim \triangle T$, and hence $\triangle T_1 \sim \triangle T'$.

(2) $\triangle T_1 = \triangle T'$.

For this show that $A_1B_1 = A'B'$ by means of the equations

$$\frac{A_1B_1}{AB} = \frac{OA_1}{OA} \text{ and } \frac{A'B'}{AB} = \frac{OA_1}{OA}.$$

Likewise $A_1C_1 = A'C'$ and $B_1C_1 = B'C'$.

(3) Finally, $$\frac{OM_1}{OM} = \frac{OA_1}{OA},$$

where M and M_1 are any two corresponding points.

Hence O is the required center of similitude.

SIMILAR TETRAHEDRONS PLACED IN SIMILITUDE

390. THEOREM VIII. *Two similar tetrahedrons may be so placed as to have a center of similitude.*
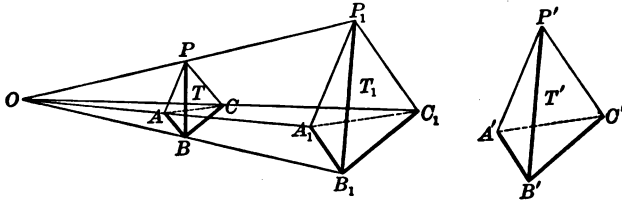
Given the similar tetrahedrons T and T'.

To prove that they can be placed so as to have a center of similitude.

Proof: With O as a center of similitude, construct T_1, making

$$\frac{OA}{OA_1} = \frac{OB}{OB_1} = \text{etc.} = \frac{AB}{A'B'}.$$

Now show as in § 389 that $T_1 = T'$, and hence that T' can be placed in the position T_1 so as to have with T the center of similitude O.

Give all the steps in detail.

391. COROLLARY. *Any two similar polyhedrons may be placed so as to have a center of similitude.*

Suggestion. The argument is precisely similar to that in § 390.

392. In the proofs for the two preceding theorems the center of similitude was taken *between* the two figures or *on the same side* of them. The center may be taken equally well *within* them, as in the following illustrations:
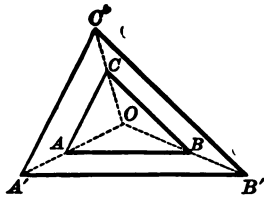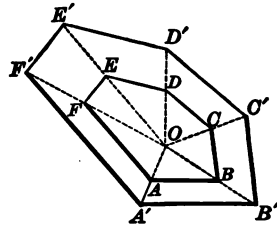
FIG. 1   FIG. 2

In the case of similar convex polyhedrons with the center of similitude thus placed, the faces are the bases of pyramids whose vertices are all at the center of similitude.

If, further, the polygonal faces be divided into triangles by drawing their diagonals, these triangles become the bases of tetrahedrons, all of whose vertices are at the center of similitude.

FIG. 3

Moreover, each inner tetrahedron is similar to its corresponding outer tetrahedron. (Why?)

The volumes of the two similar polyhedrons are thus composed of the sums of sets of similar tetrahedrons.
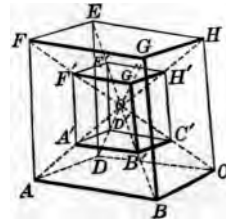
SIGHT WORK

1. Give the proof of § 389, using Fig. 1 above, and extend the argument to two similar polygons, using Fig. 2.

2. Draw two similar tetrahedrons with their center of similitude within them, and give the proof of § 390 with the figure thus made.

3. Give the proof of § 390, using Fig. 3 above.

RATIO OF VOLUMES OF SIMILAR POLYHEDRONS

393. Theorem IX. *The volumes of any two similar polyhedrons are proportional to the cubes of their corresponding edges.*

Proof: Place the polyhedrons whose volumes are V and V' so as to have their center of similitude within them as in the third figure of § 392.

Call the volumes of the similar tetrahedrons T_1, T_2, T_3, $\cdots$, and T'_1, T'_2, T'_3, $\cdots$, and let AB and $A'B'$ be two corresponding edges.

Then we have

$$\frac{\overline{AB}^3}{\overline{A'B'}^3} = \frac{T_1}{T'_1} = \frac{T_2}{T'_2} = \frac{T_3}{T'_3} = \cdots . \qquad \text{(Why?)}$$

And $\quad \dfrac{T_1 + T_2 + T_3 + \cdots}{T'_1 + T'_2 + T'_3 + \cdots} = \dfrac{T_1}{T'_1} = \dfrac{\overline{AB}^3}{\overline{A'B'}^3}. \qquad \text{(Why?)}$

But $\quad T_1 + T_2 + T_3 \cdots = V$ and $T'_1 + T'_2 + T'_3 \cdots = V'$.

Hence, $\qquad \dfrac{V}{V'} = \dfrac{\overline{AB}^3}{\overline{A'B'}^3}.$

394. Corollary. *The volumes of any two similar solids are proportional to the cubes of any two corresponding linear dimensions.*

This proposition may be rendered evident by noticing that any two similar three-dimensional figures may be built up to any degree of approximation by means of pairs of similar tetrahedrons similarly placed. The proposition then holds for any two corresponding figures used in the process of approximation.

Note that the ratio of similitude of two similar figures may be obtained from the ratio of any pair of their corresponding linear dimensions.

APPLICATIONS OF SIMILARITY

395. The Pantograph. The theorem that any two figures which have a center of similitude are similar is the geometric basis of many mechanical contrivances for enlarging or reducing both plane and solid figures; that is, for constructing figures similar to given figures and having with them a given ratio of similitude.

The essential property of all such contrivances is that one point *O* is kept fixed, while two points *A* and *B* are allowed to move so that *O*, *A*, and *B* always remain in a straight line, and so that the ratio *OA* : *OB* remains the same. See page 216 of Plane Geometry.

In the first figure on this page *O* is a fixed point. Segments *OD*, *CB*, and the sides of the parallelogram *ACED* are of fixed length.

Prove that if *B* is once so taken on the line *EC* as to be in the line *OA*, the points *O*, *A*, and *B* will always remain collinear, and that *OA* : *OB* remains a fixed ratio.

In the second figure is shown an ordinary *pantograph* used for copying and at the same time for reducing or enlarging maps, designs, etc. The lengths of the various segments are adjustable, as shown, thus obtaining any desired scale.

The same contrivance may be used for copying figures in *space*, such as relief maps, and at the same time reducing or enlarging them.
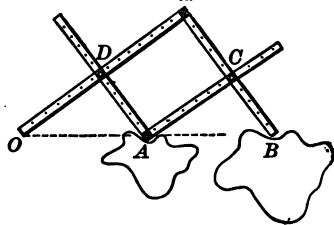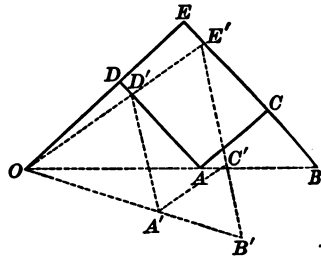
RATIOS RELATING TO ANY TWO SIMILAR SOLIDS

396. Corresponding Cross Sections. Now consider any two similar figures whatever so placed as to have a center of similitude O. We have seen that if points A, B and A', B' are corresponding points of the two figures, then the ratio of the *corresponding linear dimensions* AB and $A'B'$ is equal to the ratio of similitude $m : n$ of the two figures.

Also if A, B, C, D and A', B', C', D' are corresponding points, then $\triangle ABC$ and $A'B'C'$, and the tetrahedrons $ABCD$ and $A'B'C'D'$ are similar, and we have

$$\frac{\text{area } ABC}{\text{area } A'B'C'} = \frac{m^2}{n^2} \text{ and } \frac{\text{vol. } ABCD}{\text{vol. } A'B'C'D'} = \frac{m^3}{n^3}.$$

The points A, B, C and A', B', C' determine two planes, each of which intercepts a certain plane figure in the solid figure to which the points belong. These two plane figures we call *corresponding cross sections*.

We assume without full argument:

397. THEOREM X. (1) *The ratio of the areas of any pair of corresponding cross sections or any pair of corresponding surfaces of similar figures is equal to the square of their ratio of similitude,* and

(2) *The ratio of the volumes of any two similar figures is equal to the cube of their ratio of similitude.*

The fact that the ratio of the areas of corresponding surfaces of similar solids is equal to the *square* of their ratio of similitude, while the ratio of their volumes equals the *cube* of this ratio is one of the most important and far-reaching conclusions of geometry.

Thus the ratio of the weights of two similar shells used in gunnery, or the ratio of the weights of two men of similar build, may be found when their ratio of similitude is known.

PROBLEMS AND APPLICATIONS

1. If it is known that a steel wire of radius r will carry a certain weight w, how great a weight will a wire of the same material carry if its radius is $2r$?

Suggestion. The tensile strengths of wires are in the same ratio as their cross-section areas.

2. Find the ratio of the diameters of two wires of the same material if one carries twice the load of the other; three times the load.

3. In a laboratory experiment a heavy iron ball is suspended by a steel wire. In suspending another ball of twice the diameter a wire of twice the radius of the first one is used. Is this perfectly safe if it is known that the first wire will just safely carry the ball suspended from it? Discuss fully.

4. In two schoolrooms of the same shape (similar figures) but of different size, the same *proportion* of the floor space is occupied by desks. Which contains the larger amount of air for each pupil?

5. If the shells used in guns are similar in shape, find the ratio of the total surface areas of an eight-inch and a twelve-inch shell.

6. Find the ratio of the weights of the shells in the preceding problem, weights being in the same ratio as the volumes.

7. If a man 5 ft. 9 in. tall weighs 165 lb., what should be the weight of a man 6 ft. 1 in. tall, if they are similar in shape?

8. What is the diameter of a gun which fires a shell weighing twice as much as a shell fired from an eight-inch gun, supposing the shells to be similar bodies?

9. Supposing two trees to be similar in shape, what is the diameter of a tree whose volume is three times that of one whose diameter is 2 feet? What is the diameter if the volume is five times that of the given tree? What if it is n times that of the given tree?

10. Assuming that the weights of schoolboys vary as the cubes of their heights, construct a graph representing the relation between their heights and weights, if a boy 5 feet 9 inches tall weighs 130 pounds.

Suggestion. If w represents the number of pounds in weight and h the number of feet in height, $w = kh^3$. From $w = 130$, when $h = 5\frac{3}{4}$, we have $k = .684$. For the purpose of the graph, $k = .7$ is accurate enough.

11. From the graph constructed in the preceding example find the weight of a boy 5 feet tall; one 5 feet 4 inches; one 5 feet 6 inches. Compare with the weights of boys in your class.

12. If a man 6 feet tall weighs 185 pounds, construct a graph representing the weights of men of similar build and of various heights.

13. If steamships are of the same shape, their tonnages vary as the cubes of their lengths. The *Mauretania* is 790 feet long, with a net tonnage of 32,500. Construct a graph representing the tonnage of steamships of the same shape, and of various lengths.

Other ships which at one time or another have held ocean records are: the (former) *Deutschland,* length 686 ft. and tonnage 16,500; the (former) *Kaiser Wilhelm der Grosse,* length 648 ft. and tonnage 14,300; the *Lucania,* length 625 ft. and tonnage 13,000 (nearly); and the *Etruria,* length 520 ft. and tonnage 8000. By means of this graph decide whether or not these boats have greater or less tonnage than the *Mauretania* as compared with their lengths.

14. Raindrops as they start to fall are extremely small. In the course of their descent a great many are united to form larger and larger drops. If 1000 such drops unite into one, what is the ratio of the surface of the large drop to the sum of the surfaces of the small drops?

15. The strength of a muscle varies as its cross-section area, which in turn varies as the square of the height or length of an animal, while the weight of the animal varies as the cube of its height or length. Use these facts to explain the greater agility of small animals. For example, compare the rabbit and the elephant.

16. Assuming the velocities the same, the amounts of water flowing through pipes vary directly as their cross-section areas. How many pipes, each 4 in. in diameter, will carry as much water as one pipe 72 in. in diameter ?

17. What must be the diameter of a cylindrical conduit which will carry enough water to supply ten circular intakes each 8 feet in diameter ?

18. A water reservoir, including its feed pipes, is replaced by another, each of whose linear dimensions is twice the corresponding dimensions of the first. If the velocity of the water in the feed pipes of the new system is the same as that in the old, will it take more or less time to fill the new reservoir than it did the old ? What is the ratio of the new time to the old ?

19. If two engine plants are exactly . similar in shape, but each linear dimension in one is three times the corresponding dimension of the other, and if the steam in the feed pipes flows with the same velocity in both, compare the speeds of the engines.

20. If two men, one 5 ft. 6 in. and the other 6 ft. 2 in. in height, are similar in structure in every respect, how much faster must the blood flow in the larger person in order that the body tissues of both shall be supplied equally well ?

Suggestion. Note that the amount of tissue to be supplied varies as the cube of the height, while the cross-section area of the arteries varies as the square of the height.

APPENDIX II : PROJECTION OF LINE-SEGMENTS

398. Length of Projection. The projection of a line-segment on a plane was shown in § 121 to be another line-segment. The *length of this projection* will now be computed in terms of the given line-segment.

399. Cosine of Projection Angle. The acute angle between a line-segment and a given line on which it is projected is called the *projection angle*.

If l is the length of a line-segment and p the length of its projection, then the ratio $p : l$ is called the *cosine of the projection angle*.

E.g., in Fig. 1, $\dfrac{p}{l} =$ cosine $\angle BAE$.
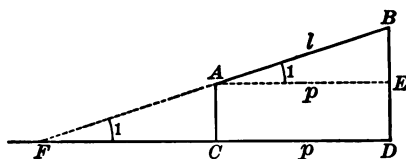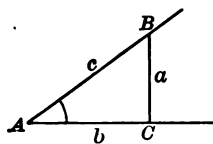
Fig. 1   **Fig. 2**

400. Sine, Cosine, Tangent. In any right triangle ABC (Fig. 2), either acute angle, as $\angle A$, is the projection angle between the hypotenuse and the side adjacent to the angle.

Hence the *cosine* of an acute angle of a right triangle is the *ratio of the adjacent side to the hypotenuse.*

Likewise we define the *sine* of an acute angle of a right triangle as the *ratio of the opposite side to the hypotenuse,* and the *tangent* of an acute angle of a right triangle as the *ratio of the opposite side to the adjacent side.*

Using the common abbreviations, sin, cos, and tan, we have in Fig. 2,

$$\sin A = \frac{a}{c}, \ \cos A = \frac{b}{c}, \ \tan A = \frac{a}{b}$$

EXERCISES ON SINES, COSINES, TANGENTS

The sine, cosine, and tangent are of great importance in many computations. By careful measurement (and in other ways) their values may be computed for any acute angle, and a table formed, like that on page 175.

E.g., if $\angle A = 35°$ (measured with a protractor), and if we measure AC, AB, and BC, and then compute the ratios $\frac{a}{c}$, $\frac{b}{c}$, and $\frac{a}{b}$, we shall find the values of sin 35°, cos 35°, tan 35°.

With an ordinary ruler it will not usually be possible to make these measurements with sufficient accuracy to obtain more than one decimal place.
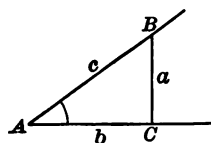
EXERCISES

1. Using a protractor, construct angles of 10°, 30°, 50°, 70°, and by measurement determine the sine, cosine, and tangent of each.

2. Prove that the cosine of any given angle is the same, no matter what point is taken in *either* side from which to let fall the perpendicular to the other side. Prove the same for the tangent.
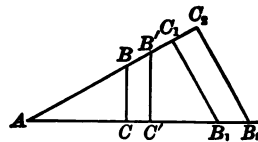
3. Show that if the hypotenuse be taken *one decimeter* in length, then the length of the side adjacent, measured in decimeters, is the *cosine* of the angle, and the length of the side opposite is the *sine* of the angle.

4. Show that if the side adjacent be taken one decimeter in length, the length of the side opposite, measured in decimeters, is the *tangent* of the angle.

5. Without any direct measurement, show how to compute the three ratios for each of the angles 30°, 45°, 60°.

Suggestion. Make use of the fact that if one acute angle in a right triangle is 30°, the side opposite it is one half the hypotenuse.

TABLE OF SINES, COSINES, AND TANGENTS

Angle	Sin	Cos	Tan	Angle	Sin	Cos	Tan
0°	0	1.000	0	46°	.719	.695	1.04
1°	.017	1.000	.017	47°	.731	.682	1.07
2°	.035	.999	.035	48°	.743	.669	1.11
3°	.052	.999	.052	49°	.755	.656	1.15
4°	.070	.998	.070	50°	.766	.643	1.19
5°	.087	.996	.087	51°	.777	.629	1.23
6°	.105	.995	.105	52°	.788	.616	1.28
7°	.122	.993	.123	53°	.799	.602	1.33
8°	.139	.990	.141	54°	.809	.588	1.38
9°	.156	.988	.158	55°	.819	.574	1.43
10°	.174	.985	.176	56°	.829	.559	1.48
11°	.191	.982	.194	57°	.839	.545	1.54
12°	.208	.978	.213	58°	.848	.530	1.60
13°	.225	.974	.231	59°	.857	.515	1.66
14°	.242	.970	.249	60°	.866	.500	1.73
15°	.259	.966	.268	61°	.875	.485	1.80
16°	.276	.961	.287	62°	.883	.469	1.88
17°	.292	.956	.306	63°	.891	.454	1.96
18°	.309	.951	.325	64°	.899	.438	2.05
19°	.326	.946	.344	65°	.906	.423	2.14
20°	.342	.940	.364	66°	.914	.407	2.25
21°	.358	.934	.384	67°	.921	.391	2.36
22°	.375	.927	.404	68°	.927	.375	2.48
23°	.391	.921	.424	69°	.934	.358	2.61
24°	.407	.914	.445	70°	.940	.342	2.75
25°	.423	.906	.466	71°	.946	.326	2.90
26°	.438	.899	.488	72°	.951	.309	3.08
27°	.454	.891	.510	73°	.956	.292	3.27
28°	.469	.883	.532	74°	.961	.276	3.49
29°	.485	.875	.554	75°	.966	.259	3.73
30°	.500	.866	.577	76°	.970	.242	4.01
31°	.515	.857	.601	77°	.974	.225	4.33
32°	.530	.848	.625	78°	.978	.208	4.70
33°	.545	.839	.649	79°	.982	.191	5.14
34°	.559	.829	.675	80°	.985	.174	5.67
35°	.574	.819	.700	81°	.988	.156	6.31
36°	.588	.809	.727	82°	.990	.139	7.12
37°	.602	.799	.754	83°	.993	.122	8.14
38°	.616	.788	.781	84°	.995	.105	9.51
39°	.629	.777	.810	85°	.996	.087	11.43
40°	.643	.766	.839	86°	.998	.070	14.30
41°	.656	.755	.869	87°	.999	.052	19.08
42°	.669	.743	.900	88°	.999	.035	28.64
43°	.682	.731	.933	89°	1.000	.017	57.29
44°	.695	.719	.966	90°	1.000	0	
45°	.707	.707	1.00				

401. THEOREM I. *The length of the projection of a line-segment upon a given line is equal to the length of the line-segment multiplied by the cosine of the projection angle.*

Given the projection p of the line-segment l on the line CD, with the projection angle A.

To prove that $p = l \cos A$.

Proof: By definition we have

$\dfrac{p}{l} = \cos A$.   Hence, $p = l \cos A$.
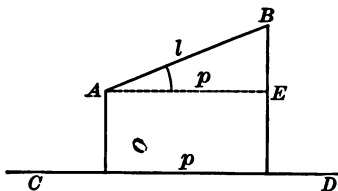
EXERCISES

1. Find the cosines of the angles 35° 30', 54° 15', 15° 45'.

Suggestion. The cosine of 35° 30' lies between cos 35° and cos 36°. We assume that it lies halfway between these numbers. This assumption, while not quite correct, is very nearly so for small differences of angles, as in this case, where the total difference is only one degree. From the table cos 35° = .819, cos 36° = .809. The number midway between these is .814, which we take as the cosine of 35° 30'.

This process is called **interpolation.** A similar process is used for sines and tangents.

2. Find the tangents of the angles 25° 20', 47° 45', 63° 40'.

3. Find the angle whose tangent is 1.74.

Solution. From the table we have tan 60° = 1.73 and tan 61° = 1.80. Hence, the required angle must lie between 60° and 61°. Moreover, the number 1.74 is one seventh the way from 1.73 to 1.80. Hence, we assume the angle to lie one seventh the way from 60° to 61°, which gives $60° + \frac{1}{7} \times 1° = 60° + 9'$ nearly. The required angle is 60° 9'.

4. Find the angles whose sines are .276; .674; .437.

5. Find the angles whose cosines are .940; .094; .435.

6. Find the angles whose tangents are .781; 1.41; 3.64.

7. At what angle with the horizontal must the base of a right circular cylinder be tilted to make it just topple over if its diameter is 6 ft. and its altitude 8 ft.?
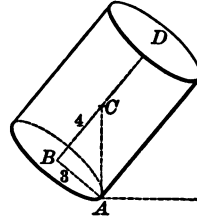
Suggestion. The center of gravity is at the middle point C of the axis of the cylinder. The base must be tilted so that the line AC becomes vertical. The required angle is $\angle ACB$.

8. The Leaning Tower of Pisa is 179 feet high and 31 feet in diameter. It now leans so that a plumb line from the top on the lower side reaches the ground 14 feet from the base. At what angle is its side now inclined from the vertical? At what angle would its side have to incline from the vertical before it would topple over?
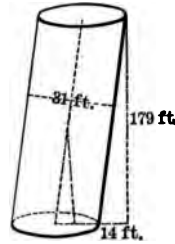
9. A four-inch hole is cut in a board, and a ball 8 in. in diameter is made to rest on it. At what angle must the board be held so that the ball will just roll out of the hole?
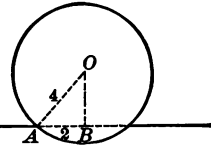
Suggestion. The board must be held so that the line OA becomes vertical.

10. Using a ball 8 inches in diameter, what must be the radius of the hole in the board of the preceding problem so that the ball shall just roll out when the board is inclined at an angle of 45° to the horizontal?

11. If the figure $ABCD$–H is a cube, find each of the following angles: $\angle ECA$, $\angle AEC$.

Check by using the fact that the sum of the angles of a triangle is 180°.
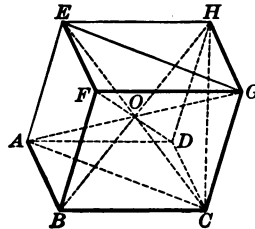
ALTITUDE OF OBLIQUE PRISM OR CYLINDER

402. THEOREM II. *The altitude of an oblique prism or cylinder is equal to an element multiplied by the cosine of the angle between the plane of the base and that of a right section.*
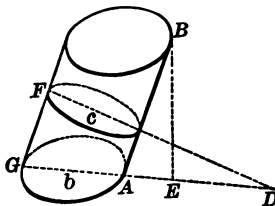
Given an oblique prism or cylinder with base b and right section c, and let BE be a perpendicular between the planes of the bases.

Consider the plane determined by BE and the element AB.

This plane is $\perp$ to the plane of b and also to the plane of c. (Why?)

Hence, it is $\perp$ to the line of intersection of the planes of b and c. (Why?)

Let this plane cut the planes of b and c in GD and FD respectively.

Then $\angle D$ is the measure of the dihedral angle between the planes of b and c. (Why?)

To prove that $BE = AB \cdot \cos D.$

Proof: We have $BE = AB \cdot \cos \angle ABE.$ Why?

But $\angle D = \angle ABE.$ Why?

Hence, $BE = AB \cdot \cos D.$

403. COROLLARY. *The dihedral angle between the planes of the base and a right section of an oblique cylinder or prism is equal to the angle between an element and the altitude.*

EXERCISES

1. Given a line-segment 10 inches long. Find the length of its projection on a plane if the projection angle is 20°. If the angle is 30°, 45°, 60°, 90°, 0°.

2. A kite string forms an angle of 40° with the ground. The distance from the end of the string to a point directly beneath the kite is 200 ft. Find the length of the string and the perpendicular height of the kite.

3. The altitude of an oblique prism is 15 inches. Find the length of an element if it makes an angle of 45° with the perpendicular between the bases.

4. A right section of a cylinder makes an angle of 20° with the plane of the lower base. Find the ratio between the altitude and an element.

5. Prove that by joining the middle points of six edges of a cube, as shown in the figure, a regular hexagon is formed.

6. Prove that in the preceding example the plane of the regular hexagon, *KLMNOP*, is perpendicular to the diagonal *DF* of the cube.
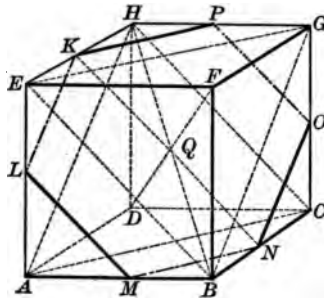
7. How large a cube will be required from which to cut a stopper for a hexagonal spout, each of whose sides is 4 inches ?

8. In the figure find the angle *KQH*.

Suggestion. Let a be a side of the cube. Compute *KH*, *KQ*, and *HQ* in terms a. Note that $\angle QKH = $ rt. $\angle$.

9. Find the area of the projection of the hexagon *KLMNOP* on the face *BCGF*. Note that this projection equals the whole square less $\triangle NCO + \triangle KEL$. See § 404. Find the area of the hexagon in terms of the side a of the cube.

404. Projection of a Plane-Segment. If from each point in the boundary of a plane-segment a perpendicular is drawn to a given plane, the locus of the feet of these perpendiculars will bound a portion of the plane, which is called the *projection of the plane-segment* on the given plane.

E.g., the plane-segment $A'B'C'$ in the plane N is the projection of the plane-segment ABC from the plane M upon N.

The *angle of projection* is the angle between the planes M and N.
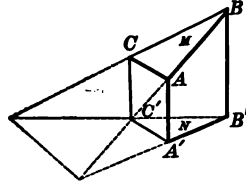
405. THEOREM III. *The area of the projection of a plane-segment on a plane is equal to the area of the plane-segment multiplied by the cosine of the projection angle.*
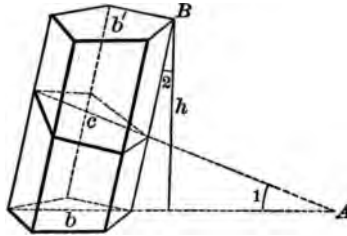
Proof: Let the boundary of the given plane-segment b be any convex polygon or closed curve.

Using a line perpendicular to the given plane as a *generator*, develop a prismatic or cylindrical surface of which b is a section. The given plane will cut this surface in a right section whose area we denote by c.

Now cut the surface by a plane parallel to b, forming the upper base b' of a prism or cylinder whose altitude is h, edge e, and volume V.

Then c is the projection of b upon the given plane, and $\angle 1 = \angle 2$ is the projection angle.

We are to show that $c = b \cos \angle 1$.

We know that $\qquad V = ce = bh.$ $\qquad\qquad$ **Why?**

But $\qquad\qquad h = e \cos \angle 2.$

Hence, $\qquad\qquad ce = be \cos \angle 2.$

That is, $\qquad\qquad c = b \cos \angle 2 = b \cos \angle 1.$

Note. The foregoing theorem may be proved *directly* in case the plane-segment is a rectangle with one side parallel to the line of intersection of the two planes. In the figure let S be the given rectangle and S' its projection, with $AB \parallel$ to the line of intersection of the planes in which S and S' lie, and $\angle 1$ the angle between them.

Then $\qquad S = AB \cdot BC$

and $\qquad S' = A'B' \cdot B'C'.$

But $\qquad AB = A'B'$

and $\qquad BE = B'C'.$ $\qquad$ **Why?**

But $\qquad BE = BC \cdot \cos \angle 1$ $\qquad$ **Why?**

and $\qquad S' = A'B' \cdot B'C' = AB \cdot BC \cos \angle 1.$

That is, $\qquad S' = S \cos \angle 1.$
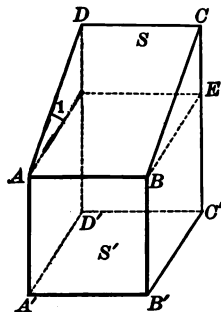
In the case of any plane-segment, rectangles may be inscribed in it in this position and their number increased indefinitely, so that their sum will approach more and more nearly to the area of the plane-segment, and in this way it may be shown to any desired degree of approximation, that the projection of a plane-segment equals the given plane-segment multiplied by the cosine of the projection angle.

406. Ellipse. An important special case of the theorem of § 405 is the area of the figure obtained by projecting a circle upon a plane not parallel to its own plane, nor at right angles to it. This figure is called an *ellipse*. On page 182 the principle developed above is used to find the area of the ellipse.

407. Projection of a Circle.   In the figure two planes, M and M', meet· in a line PQ. The circle O in M has a diameter $AB \parallel PQ$ and a diameter $CD \perp PQ$.

In projecting the whole figure upon the plane M' the diameter AB projects into its equal $A'B'$, while CD projects into $C'D'$ so that $C'D' = CD \cos \angle 1$.
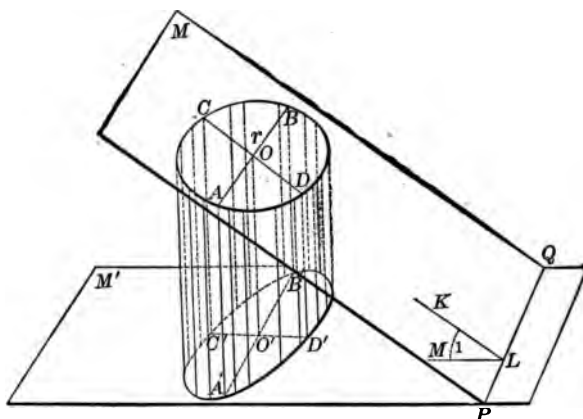
By theorem § 405 the area of the ellipse $A'C'B'D'$ equals the area of the circle $ACBD$ multiplied by $\cos \angle 1$.

Hence, $\pi r^2 \cdot \cos \angle 1$ is the area of the ellipse.

But $r \cos \angle 1 = O'C'$ and $r = O'B'$.   (§ 401)

Hence, the area of the ellipse is $\pi \cdot O'C' \times O'B'$.

The segments $A'B'$ and $C'D'$ are called respectively the major and minor axes of the ellipse, and $O'B'$ and $O'C'$ the semimajor and the semiminor axes.   These latter are usually denoted by a and b.

Hence,   **the area of the ellipse is πab.**

Note that when a and b are equal, the ellipse becomes a circle, and this formula reduces to πa^2 as it should.

PROBLEMS AND APPLICATIONS

1. Given a right circular cylinder the radius of whose base is 6 inches. Find the area of an oblique cross section inclined at an angle of 45° to the plane of the base.

2. Given an oblique circular cylinder the radius of whose right section is 10 inches. Find the area of the base if it is inclined at an angle of 60° to the right section.

3. If an oblique circular cylinder has an altitude h, an element e, radius of right section r, and $\angle A$ the inclination of the base to the right section, express the volume in two ways and show that these are equivalent.

4. A six-inch stovepipe has a 45° elbow angle, that is, it turns at right angles. (The angle CAB is called the elbow angle.) Find the area of the cross section of AB. Likewise if it has a 60° elbow angle.
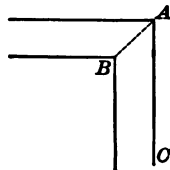
5. At what angle must the damper in a circular stovepipe be turned in order to obstruct just half the right cross sectional area of the pipe?

Suggestion. The damper must be turned so that the projection of the damper upon a right cross section is equal to half that cross section.

6. The comparatively low temperature of the earth's surface near the pole, even in summer, when the sun does not set for months, is due partly to the *obliqueness* with which the sun's rays strike the earth. That is, a given amount of sunlight is spread over a larger area than in lower latitudes.

Thus, if in the figure $D'C$ is a horizontal line, and $D'D$ the direction of the sun's rays, then a beam of light whose right cross section is $ABCD$ is spread over the rectangle $A'BCD'$. In other words, a patch of ground $A'BCD'$ receives only as much sunlight as a patch the size of $ABCD$ receives when the sun's rays strike it vertically. $ABCD = A'BCD' \cos \angle 1$.
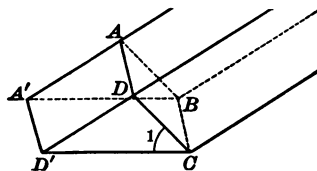
Hence, each unit of area in $A'BCD'$ receives $\cos \angle 1$ times as much light as a unit in $ABCD$.

Hence, to compare the heat-producing powers of sunlight in any latitude with that at the place where the sun's rays fall vertically, we need to know how the projection angle, $\angle 1$, is related to the difference in latitude of the two places.

7. If $\angle 1 = 30°$, compare the amount of heat received by a unit of area in $ABCD$ and $A'BCD'$.

8. What must $\angle 1$ be in order that a unit of area in $A'BCD'$ shall receive only $\frac{1}{2}$ as much light as a unit in $ABCD$?

9. The figure represents a cross section of the earth with an indication of the direction of the rays of light as they strike it at the summer solstice when they are vertical at A, the tropic of Cancer. B represents the latitude of Chicago, C the polar circle, and P the north pole. The angles PDE, CGF, BKH represent the projection angle, $\angle 1$, for the various latitudes. Prove:

$\angle PDE = \angle POK$,
$\angle CGF = \angle COK$,
$\angle BKH = \angle BOK$.

That is, $\angle 1$ for each place is the latitude of that place minus the latitude of the place where the sun's rays are vertical.
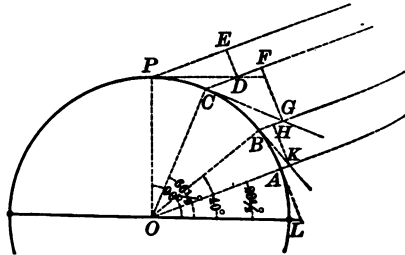
10. Find the relative amount of sunlight received by a unit of area at the tropic of Cancer and at the north pole at the time of the summer solstice.

11. Find the ratio between the amount of light received by a unit of the earth's area at Chicago and at the tropic of Cancer at the time of the summer solstice.

12. Find the same ratio for the equator and Chicago at the winter solstice when the sun is vertical at latitude $23\frac{1}{2}°$ south.

APPENDIX III: VARIABLES. LIMITS

408. Variables and Functions. It is often useful to think of a geometric figure as *varying* in size and shape.

E.g., if a rectangle has a fixed base, say 10 inches long, but an altitude which varies from 3 inches to 5 inches, then the area varies from $3 \cdot 10 = 30$ to $5 \cdot 10 = 50$ square inches.

We may even think of the altitude as starting at zero inches and increasing, in which case the area starts at zero and increases continuously.

The altitude which we think of as varying at our pleasure is called the *independent variable*, while the area, being dependent upon the altitude, is called the *dependent variable*.

The dependent variable is sometimes called a *function* of the independent variable, meaning that the two are connected by a definite relation such that, for any definite value of the independent variable, the dependent variable also has a definite value.

Thus, in the formula for the area of a rectangle, $a = bh$, if b is fixed and h varies, then a is a function of h, since for every value of h there is determined a definite value of a.

409. Illustrations of Limits. If a regular polygon is inscribed in a circle of fixed radius, and if the number of sides of the polygon be continually increased, for instance by repeatedly doubling the number, then the apothem, perimeter, and area are all variables depending upon the number of sides. That is, each of these is a function of the number of sides.

Now the greater the number of sides the more nearly does the apothem equal the radius in length. Indeed, it is evident that the difference between the apothem and the radius will ultimately become less than any fixed number, however small. Hence we say that the apothem *approaches the radius as a limit* as the number of sides increases indefinitely.

Similarly the perimeters of the polygons may be made as nearly equal to the circumference as we please by making the number of sides sufficiently great.

Hence we may define the circumference of a circle as the *limit of the perimeter* of a regular inscribed polygon as the number of sides increases indefinitely.

The circumference of a circle may also be defined as the limit of the perimeter of a circumscribed polygon as the number of sides is increased indefinitely.

Likewise we may define the area of a circle as the *limit of the area* of the inscribed or the circumscribed polygon as the number of sides is increased indefinitely.

The notion of a limit may be used to define the length of a line-segment which is incommensurable with a given unit segment.

Thus, the diagonal d of a square whose side is unity is $d = \sqrt{2}$. Hence d may be defined as the *limit of the variable line-segment* whose successive lengths are 1, 1.4, 1.41, 1.414, ⋯.

In like manner, the length of any line-segment, whether commensurable or incommensurable with the unit segment, may be defined in terms of a limit.

Thus, if a variable segment is increased by successively adding to it one half the length previously added, then the segment will approach a limit. If the initial length is 1, and if the successive additions are $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$, etc., then the successive lengths are 1, $1\frac{1}{2}$, $1\frac{3}{4}$, $1\frac{7}{8}$, $1\frac{15}{16}$, $1\frac{31}{32}$, etc. Evidently this segment approaches the limit 2.

Hence 2 may be defined as the limit of the variable segment whose successive lengths are 1, $1\frac{1}{2}$, $1\frac{3}{4}$, $1\frac{7}{8}$, etc., as the number of successive additions is increased indefinitely.

The idea of a functional relation between variables and the idea of a limit, as illustrated above, are two of the most important concepts in all mathematics.

DEFINITION OF A LIMIT

410. Constant. A quantity which remains fixed in value throughout a discussion is called a *constant*.

E.g., the base of the rectangle mentioned in § 408 is a constant.

411. Variable. A quantity which continuously changes in value, or which takes on a succession of different values, is called a *variable*.

E.g., the altitude and the area mentioned in § 408 are variables.

412. Limit of a Variable. If a variable may be made to approach a certain constant quantity in such a way that the difference between the constant and the variable becomes and remains less than any assignable value, however small, then the constant is called the *limit of the variable*.

E.g., the fixed circumference of the circle is the limit of the variable perimeters of the polygons mentioned in § 409.

SIGHT WORK

1. Find the limit of a variable line-segment whose initial length is 6 inches, and which varies by successive increments each equal to one half the preceding, the first increment being 2 inches.

2. Construct a right triangle whose sides are 1 and 2. By approximating a square root, find five successive lengths of a segment which approaches the length of the hypotenuse as a limit.

3. If one tangent to a circle is fixed and another is made to move so that their intersection point approaches the circle, what is the limiting position of the moving tangent ? What is the limit of the measure of the angle formed by these tangents ?

4. The arc *AB* of 74° is the greater of the two arcs intercepted between two secants meeting at *C* outside the circle. The points *A* and *B* remain fixed while *C* moves up to the circle. What angle is the limit of the variable angle formed by the varying secants ? What is the limit of the measure of this angle ?

5. If in the preceding the secants meet within the circle, what is the limit of their angle and also of the measure of this angle as the intersection point moves up to the center of the circle ?

THE INCOMMENSURABLE CASES

413. The Incommensurable Cases. We have seen that there are segments which are *incommensurable;* that is, which have no common unit of measure, — for instance, the side and the diagonal of a square.

For *practical purposes* the lengths of such segments are approximated to any desired degree of accuracy, and their ratios are understood to be the ratios of these approximate numerical measures.

But for *theoretical purposes* it is important to consider these incommensurable cases further; just as in algebra we not only *approximate* such roots as $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, etc., but we also deal with these surds as *exact* numbers.

Instances of this kind occur in such operations as

$$(\sqrt{3} + \sqrt{2})(\sqrt{3} - \sqrt{2}) = 3 - 2 = 1.$$

While the length of the diagonal of a unit square cannot be expressed as an integer or as a rational fraction, that is, as the quotient of two integers, we nevertheless think of such a segment as having a *definite length*, or what is the same thing, a *definite ratio* with the unit segment forming the side of the square.

We now fix our attention on the *incommensurable ratios themselves*, and the method of determining them, rather than on the *process of approach* and the practical computation based on it.

414. Representation by Irrational Numbers. In general, the ratio between any two incommensurable geometric magnitudes of the same kind may be represented by what is called an *irrational number;* that is, a number which is *neither an integer nor a quotient of two integers.*

Examples of irrational numbers are the surds, such as $\sqrt{2}$ $\sqrt[3]{5}$, etc., and the number π.

NUMBERS DEFINED BY SEQUENCES

415. Numbers Defined by Sequences. The following is a method for determining any number, whether rational or ‘irrational. For simplicity it is applied first to the integer 1.

Throughout this discussion the expressions " point on a line " and " number " will be used interchangeably.

On a straight line mark a certain point 0 (zero), and one unit to the right of it mark another point 1. Also lay off points such that their distances from 0 are, $\frac{1}{2}$, $\frac{3}{4}$, $\frac{7}{8}$, $\frac{15}{16}$, ⋯.
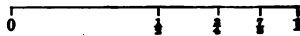
If this sequence of points is carried ever so far, it will never reach the point 1. If, however, we select a point to the left of 1, no matter how near to it, we may always go far enough along this sequence to reach points between this point and 1. That is, 1 is the limit of the terms of this sequence. See § 412.

The point 1 has two definite relations to this sequence:

(*a*) *Every point of the sequence is to the left of* 1.

(*b*) *For any fixed point to the left of* 1 *there are points of the sequence between it and* 1.

We see that 1 is the *only* point on the whole line such that both (*a*) and (*b*) are true of it. For every point to the right of 1 (*a*) is true, but (*b*) is not. For every point to the left of 1 (*b*) is true, but (*a*) is not.

It follows therefore that, while the points of the sequence merely *approach* the point 1 as a limit, the sequence, *taken as a whole, serves to distinguish that point from all other points on the line.*

In the above sequence, the terms continually *increase*. The number 1 may be determined equally well by a *decreasing* sequence. For instance, the sequence $1\frac{1}{2}$, $1\frac{1}{4}$, $1\frac{1}{8}$, $1\frac{1}{16}$, ⋯ has the following properties which distinguish the point 1 from all other points on the line.

(*a*) *Every point of the sequence is to the right of* 1.

(*b*) *For any fixed point to the right of* 1 *there are points of the sequence between it and* 1.

LIMIT OF AN INFINITE SEQUENCE

416. Infinite Sequences. An endless sequence of either kind just described is called an *infinite sequence*.

Not every infinite sequence serves to single out a definite point in the manner shown above. Thus the sequence 1, 2, 3, 4, ⋯ fails to do so, because its terms grow large beyond all bound. Such sequences are said to be *unbounded*, while the sequences $\frac{1}{2}$, $\frac{3}{4}$, $\frac{7}{8}$, ⋯ and $1\frac{1}{2}$, $1\frac{1}{4}$, $1\frac{1}{8}$, ⋯ are *bounded*.

Again, the sequence 1, 2, 1, 2, 1, 2, ⋯ fails to single out a definite point. This sequence is said to be *oscillating*, since its terms increase, then decrease, then increase, etc., while $\frac{1}{2}$, $\frac{3}{4}$, $\frac{7}{8}$, ⋯ and $1\frac{1}{2}$, $1\frac{1}{4}$, $1\frac{1}{8}$, ⋯ are *non-oscillating*.

417. Limit of a Sequence. The number 1 is said to be the *least upper bound* of the sequence $\frac{1}{2}$, $\frac{3}{4}$, $\frac{7}{8}$, ⋯. That is, 1 is the *smallest number* beyond which the sequence does not go. 1 is also said to be the *limit of the sequence*. See § 412.

Similarly, 1 is the *greatest lower bound* or the *limit* of the sequence $1\frac{1}{2}$, $1\frac{1}{4}$, $1\frac{1}{8}$, ⋯; that is, the *greatest number* such that the sequence contains no number less than it.

In the manner described above any integer may be determined by an increasing or a decreasing infinite sequence.

E.g., the number 2 is the limit of the increasing sequence 1, $1\frac{1}{2}$, $1\frac{3}{4}$, $1\frac{7}{8}$, $1\frac{15}{16}$, ⋯, or of the decreasing sequence $2\frac{1}{2}$, $2\frac{1}{4}$, $2\frac{1}{8}$, $2\frac{1}{16}$, ⋯.

Two different sequences, both increasing or both decreasing, may also define the same number.

E.g., each of the increasing sequences $\frac{1}{2}$, $\frac{3}{4}$, $\frac{7}{8}$, $\frac{15}{16}$, ⋯ and $\frac{2}{3}$, $\frac{4}{5}$, $\frac{6}{7}$, $\frac{8}{9}$, ⋯ determines the number 1.

We notice, however, that no matter what definite number we select in either of these sequences, there is a number in the other greater than it; that is, neither sequence contains a number greater than every number in the other. See § 420.

It is apparent that any number may be defined as the limit of a sequence, and we shall assume the converse, namely, that any bounded increasing or decreasing sequence has a limit.

FUNDAMENTAL PRINCIPLES OF LIMITS

418. Axiom. We now assume that: *Every bounded increasing sequence has a least upper bound, and every bounded decreasing sequence has a greatest lower bound.*

This axiom may also be stated:

Every bounded increasing or decreasing sequence has a limit.

This axiom simply means that every such sequence singles out a definite number, rational or irrational, in the manner discussed on page 190.

Thus, if we attempt to approximate the square root of 2, we obtain a sequence, 1, 1.4, 1.41, 1.414, 1.4142, $\cdots$, having for its limit a *definite number* represented by $\sqrt{2}$, which corresponds to the length of the diagonal of a square whose side is unity.

Again, if we attempt to approximate the value of π, we obtain a sequence 3, 3.1, 3.14, 3.141, 3.1415, $\cdots$, having for its limit the definite number represented by π, which is the ratio of the circumference to the diameter of a circle.

419. Theorem I. *Two bounded sequences have the same limit if they are equal term by term.*

Proof: If the sequences are $a_1, a_2, a_3, \cdots$ and $b_1, b_2, b_3, \cdots$, and if $a_1 = b_1$, $a_2 = b_2$, $a_3 = b_3$, $\cdots$, then they are one and the same sequence and hence have the same limit by § 418.

Historical Note. The rigorous treatment of limits has been developed during the last fifty years. The foundation on which it rests is the theory of irrational numbers, which itself was placed on a firm basis in the early seventies.

The Greeks dealt with the incommensurable cases in an interesting manner. Thus, to prove that two incommensurable ratios are equal they showed that neither can be less than the other. However, they failed to make explicit definitions of incommensurables and they did not explicitly state the fundamental axiom of § 418 or its equivalent. Their treatment would be complete and rigorous if supplemented by the proper definitions and axioms.

FUNDAMENTAL PRINCIPLES OF LIMITS

420. THEOREM II. *Two increasing bounded sequences have the same limit if neither sequence contains a number greater than every number of the other.*

Let a_1, a_2, a_3, ⋯ and b_1, b_2, b_3, ⋯ denote two infinite sequences with limits a and b, such that no term of the first sequence is greater than every term of the second, and no term of the second is greater than every term of the first.

To prove that $a = b$.

Proof: Suppose that a is not equal to b but is less than b. Then there must be numbers of the sequence b_1, b_2, b_3, ⋯ greater than a, since by § 415 there are numbers b_1, b_2, b_3, greater than any fixed number whatever which is less than b. This contradicts the hypothesis of the theorem, and hence a cannot be less than b. In like manner we show that b is not less than a. Hence $a = b$. See § 20.

421. THEOREM III. *Two decreasing bounded sequences have the same limit if neither sequence contains a number less than every number of the other.*

Proof: The proof is exactly similar to that of Theorem II.

EXERCISES

1. Show that the sequences $\frac{1}{2}$, $\frac{3}{4}$, $\frac{7}{8}$, ⋯ and $\frac{3}{2}$, $\frac{5}{4}$, $\frac{9}{8}$, ⋯ have the same limits.

2. Show that the sequences $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, ⋯ and $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$, ⋯ have the same limits.

The applications of the theory of limits to geometry consist chiefly in showing that two numbers are equal because they are the limits of the same sequence, or of sequences having the property stated in the theorem of § 420 or of § 421.

APPLICATION OF LIMITS TO PLANE GEOMETRY

422. Problem. *On a given segment AB to lay off a sequence of points B_1, B_2, B_3, $\cdots$, of which B is the limit, such that each of the segments AB_1, AB_2, AB_3, $\cdots$ is commensurable with a given segment CD.*
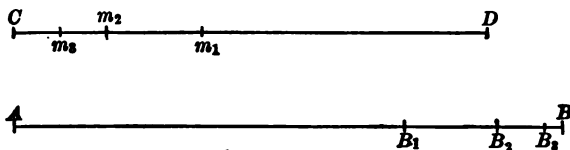
Solution: Using m_1, an exact divisor of CD, as a unit of measure, lay off on AB a segment AB_1, an exact multiple of m_1, such that the remainder B_1B is less than m_1. Then CD and AB_1 are commensurable.

Using a unit m_2, likewise a divisor of CD, and less than B_1B, lay off AB_2 such that B_2B is less than m_2. Then $AB_2 > AB_1$, and CD and AB_2 are commensurable.

Continuing in this manner, using as units of measure segments m_3, m_4, $\cdots$ each an exact divisor of CD and each less than B_2B, B_3B, $\cdots$ respectively, we obtain a sequence of segments AB_1, AB_2, AB_3, $\cdots$, each greater than the preceding and each commensurable with CD.

If the units m_1, m_2, m_3, $\cdots$ are so selected that they approach zero as a limit, it follows that B is the limit point of the sequence B_1, B_2, B_3, $\cdots$.

If a different sequence of divisors of CD, as m_1', m_2', m_3', $\cdots$, is used, we obtain a sequence of points B_1', B_2', B_3', $\cdots$, likewise satisfying the conditions of the problem.

Any two such sequences of points B_1, B_2, B_3, $\cdots$ and B_1', B_2', B_3', $\cdots$ determined as above, are both *increasing* and each is such that *no* point of it is to the right of *every* point of the other.

Hence, by § 420 any two such sequences have the same limit point B.

423. Limit of a Sequence of Segments. If B_1, B_2, B_3, $\cdots$ is a sequence of points on the segment AB having the limit B, then the segment AB is said to *be the limit of the sequence of segments* AB_1, AB_2, AB_3, $\cdots$.

INCOMMENSURABLE RATIOS

424. Ratio of Two Incommensurable Segments. The ratio of two incommensurable segments has not been explicitly defined, but this may now be done in terms of the *limit of a sequence.*

Consider two incommensurable segments AB and CD. Let $a_1, a_2, a_3, \cdots$ be the lengths of segments each commensurable with CD, forming a sequence whose limit is the segment AB, and let b be the length of the segment CD.

Then $\dfrac{a_1}{b}, \dfrac{a_2}{b}, \dfrac{a_3}{b}, \cdots$ is an increasing bounded sequence having a limit which we call R.

If $a_1', a_2', a_3', \cdots$ are the lengths of segments forming any other sequence whose limit is AB, the sequence $\dfrac{a_1'}{b}, \dfrac{a_2'}{b}, \dfrac{a_3'}{b}, \cdots$ is another increasing bounded sequence with limit R'.

By § 420 we now know that $R = R'$.

This number R is defined as *the ratio of the incommensurable segments AB and CD.*

425. Ratio of Two Incommensurable Arcs. The considerations of §§ 422, 423, 424 apply to two arcs of equal circles.
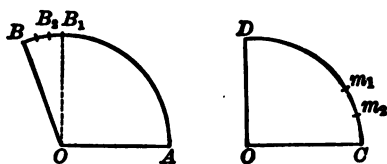
Thus, in the figure, the ratio of the two incommensurable arcs AB and CD is the limit of the sequence $\dfrac{AB_1}{CD}, \dfrac{AB_2}{CD}, \dfrac{AB_3}{CD}, \cdots$, and the ratio of the incommensurable angles AOB and COD is the limit of the sequence $\dfrac{\angle AOB_1}{\angle COD}, \dfrac{\angle AOB_2}{\angle COD}, \dfrac{\angle AOB_3}{\angle COD}, \cdots$.

APPLICATIONS OF LIMITS TO PLANE GEOMETRY

426. THEOREM IV. *A line parallel to the base of a triangle, and meeting the other two sides, divides them in the same ratio.*

Given the $\triangle ABC$ **with** $DE \parallel BC$ **and cutting** AB **and** AC.

To prove that $\dfrac{AD}{AB} = \dfrac{AE}{AC}$.

Proof: Consider the case when AD and AB are incommensurable.

Let $D_1, D_2, D_3, \cdots$ be a sequence of points on AB whose limit is D. Through these points draw parallels to BC, meeting AC in $E_1, E_2, E_3, \cdots$.
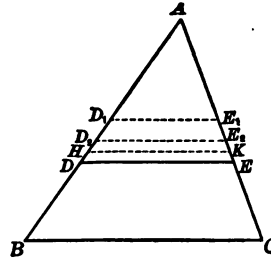
Then E is the limit of the sequence of points $E_1, E_2, E_3, \cdots$.

For suppose it is not, and that there is a point K on AE such that there is no point of $E_1, E_2, E_3, \cdots$ between K and E. Draw a line parallel to BC through K, meeting AB in H.

But there are points of the sequence D_1, D_2, D_3, *between* H and D, and hence there must be points of the sequence $E_1, E_2, E_3, \cdots$ *between* K and E, which shows that E is the limit of the sequence $E_1, E_2, E_3, \cdots$.

Then the sequence (1) $\dfrac{AD_1}{AB}, \dfrac{AD_2}{AB}, \dfrac{AD_3}{AB}, \cdots$ has a limit R which, by definition, is the ratio of the segments AD and AB.

§ 424

Similarly, the sequence (2) $\dfrac{AE_1}{AC}, \dfrac{AE_2}{AC}, \dfrac{AE_3}{AC}, \cdots$ has a limit R' which is the ratio of AE to AC.

Now $\dfrac{AD_1}{AB} = \dfrac{AE_1}{AC}, \dfrac{AD_2}{AB} = \dfrac{AE_2}{AC}, \dfrac{AD_3}{AB} = \dfrac{AE_3}{AC}, \cdots$.   § 321, P. G.

Hence, the two sequences, (1) and (2), are identical and have the same limit (§ 419). Hence $R = R'$.   Q. E. D.

APPLICATIONS TO PLANE GEOMETRY

427. THEOREM V. *In the same circle or in equal circles the ratio of two central angles is the same as the ratio of their intercepted arcs.*
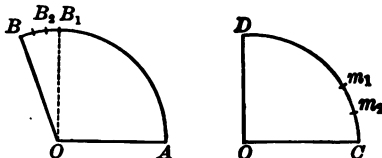
Proof : In case the arcs are commensurable the proof is obvious. For, in that case, the common measure of the arcs divides each into a number of equal parts, say m and n respectively, and these arcs subtend the same number of equal angles at the center (§ 244, Plane Geometry.) Hence the ratio of the arcs is $m : n$ and the ratio of the angles is also $m : n$.

If the arcs AB and CD are not commensurable, let AB_1, AB_2, AB_3, ··· be a sequence of arcs whose limit is AB, each arc being commensurable with the arc CD.

Then the sequence $\dfrac{AB_1}{CD}$, $\dfrac{AB_2}{CD}$, $\dfrac{AB_3}{CD}$, ··· has a limit $R = \dfrac{AB}{CD}$.

Similarly the sequence $\dfrac{\angle AOB_1}{\angle COD}$, $\dfrac{\angle AOB_2}{\angle COD}$, $\dfrac{\angle AOB_3}{\angle COD}$, ··· has a limit $R' = \dfrac{\angle AOB}{\angle COD}$.

But $\dfrac{AB_1}{CD} = \dfrac{\angle AOB_1}{\angle COD}$, $\dfrac{AB_2}{CD} = \dfrac{\angle AOB_2}{\angle COD}$, ···,

as shown above for the commensurable case.

Hence, these two sequences are identical and have the same limit (§ 419). Therefore it follows that $R = R'$.   Q. E. D.

Note. — The student should note the successive steps in the proofs of §§ 426, 427 : (*a*) Definitions of the incommensurable ratios by means of infinite sequences ; (*b*) Proof that these sequences are identical ; (*c*) Conclusion that the limiting ratios are equal.

428. Products of Irrational Numbers. If a_1, a_2, a_3, $\cdots$ is a sequence with limit a, then ka_1, ka_2, ka_3, $\cdots$ is a sequence whose limit is *defined* as ka where k is any number.

If a_1, a_2, a_3, $\cdots$ and b_1, b_2, b_3, $\cdots$ are two sequences with limits a and b respectively, then a_1b_1, a_2b_2, a_3b_3, $\cdots$ is a sequence whose limit is *defined* as ab.

Similarly if a_1, a_2, a_3, $\cdots$, b_1, b_2, b_3, $\cdots$ and c_1, c_2, c_3, $\cdots$ are sequences with limits a, b, c, then abc is defined as the limit of the sequence $a_1b_1c_1$, $a_2b_2c_2$, $a_3b_3c_3$, $\cdots$.

For a complete treatment it would be necessary to show that these definitions of multiplication of irrational numbers are consistent with the rest of arithmetic and also that these new sequences are such as to determine definite limits. This is possible but is beyond the scope of this book.

429. Area of a Rectangle. If the sides of a rectangle are incommensurable with the unit segment, we *define its area* as, follows:

Let a_1, a_2, a_3, $\cdots$ be a sequence of rational numbers whose limit is the altitude a; and let b_1, b_2, b_3, $\cdots$ be a sequence whose limit is the base b.

Then the *area of the rectangle is the limit of the sequence* a_1b_1, a_2b_2, a_3b_3, $\cdots$.
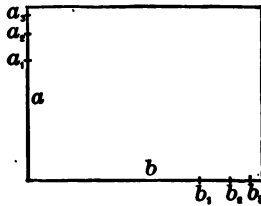
But by definition the limit of a_1b_1, a_2b_2, a_3b_3, $\cdots$ is the product ab. Hence we have the theorem:

430. Theorem VI. *The area of a rectangle is the product of its base and altitude.*

431. Incommensurable Ratios Related to Two Circles. In a circle inscribe a sequence P_1, P_2, P_3, $\cdots$ of regular polygons, each having twice the number of sides of the one preceding it.

Let the perimeters of these polygons be p_1, p_2, p_3, $\cdots$ and their areas A_1, A_2, A_3, $\cdots$. Then the length c of the circle is defined to be the limit of the sequence p_1, p_2, p_3, $\cdots$ and its area A is defined to be the limit of A_1, A_2, A_3, $\cdots$.

The sequence of polygons thus inscribed is called an *approximating sequence of polygons.*

That these sequences are *increasing* and *bounded* is obvious from the figure.
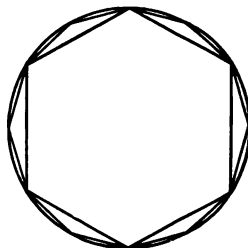
432. THEOREM VII. *The lengths of two circles are in the same ratio as their radii, and their areas are in the same ratio as the squares of their radii.*

Proof: Let the radii of the two circles be r and r'. Denote the ratio $r' : r$ by k. Then $r' = kr$.

Inscribe in one circle an approximating sequence of polygons with perimeters p_1, p_2, p_3, $\cdots$ and areas A_1, A_2, A_3, $\cdots$. In the other circle inscribe a sequence of similar polygons. By §§ 476, 477 (Plane Geometry), the perimeters of the latter are kp_1, kp_2, kp_3, $\cdots$ and their areas are k^2A_1, k^2A_2, k^2A_3, $\cdots$.

By § 428, if the limit of p_1, p_2, p_3, $\cdots$ is c and the limit of A_1, A_2, A_3, $\cdots$ is A, then the limits of kp_1, kp_2, kp_3, $\cdots$, and k^2A_1, k^2A_2, k^2A_3, $\cdots$ are kc and k^2A, respectively.

That is, the ratio of the lengths of the circles is $\dfrac{kc}{c} = k = \dfrac{r'}{r}$ and the ratio of their areas is $\dfrac{k^2A}{A} = k^2 = \dfrac{r'^2}{r^2}$.

APPLICATIONS OF LIMITS TO SOLID GEOMETRY

433. Incommensurable Solids. The areas of curved surfaces, the volumes of cones, cylinders, and spheres, and even of rectangular solids involve *incommensurable quantities*. We have already treated these cases informally and have devised means for computing them *approximately*.

We now give a formal treatment for logical completeness.

434. Rectangular Parallelopiped. If the three concurrent edges a, b, c of a rectangular parallelopiped are incommensurable with the unit segment, the volume inclosed is defined as follows:

Let a_1, a_2, a_3, $\cdots$ be a sequence of rational numbers whose limit is the side a. Let b_1, b_2, b_3, $\cdots$ and c_1, c_2, c_3, $\cdots$ be similar sequences whose limits are respectively the dimensions b and c.

Then the *volume is the limit of the sequence* $a_1 b_1 c_1$, $a_2 b_2 c_2$, $a_3 b_3 c_3$, $\cdots$.

But the limit of this sequence is by definition the product of the limits of the three sequences a_1, a_2, a_3, $\cdots$, b_1, b_2, b_3, $\cdots$, c_1, c_2, c_3, $\cdots$, or abc.

Hence, we have the theorem:

435. THEOREM VIII. *The volume of a rectangular parallelopiped is equal to the product of its three concurrent edges.*
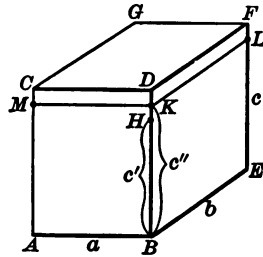
EXERCISES

1. Give in decimals four terms of a sequence which approximates the volume of a rectangular parallelopiped whose dimensions are 2, 4, $\sqrt{5}$.

2. Give in decimals four terms of a sequence which approximates the circumference of a circle whose radius is 10 in.; also of a sequence which approximates the area.

APPLICATION OF LIMITS TO THE PYRAMID

436. Triangular Pyramid. In a triangular pyramid inscribe a set of prisms in a manner similar to that shown in the figure of § 243.

Call V_1 the sum of the volumes of these inscribed prisms.

Using as altitudes one half the altitudes of the first set of prisms, inscribe a second set the sum of whose volumes is V_2. Continuing in this manner, we obtain a sequence of sets of prisms with volumes $V_1, V_2, V_3, \cdots$. The limit V of this sequence we define as **the volume of the pyramid.**

Circumscribed prisms may also be used for defining the volume of a pyramid, in which case we get a *decreasing* sequence of volumes $U_1, U_2, U_3, \cdots$, with limit U. That $U = V$ follows from the fact that the sum of the volumes of the circumscribed prisms exceeds that of the inscribed prisms by the volume of the lowest circumscribed prism, and this may be made as small as we please.

437. THEOREM IX. *Two pyramids with the same altitudes and equivalent bases have equal volumes.*

Proof: Call a sequence of inscribed prisms in one pyramid $V_1, V_2, V_3, \cdots$, and in the other $V_1', V_2', V_3', \cdots$. Since corresponding sets of these prisms have equal volumes (§ 200), that is, $V_1 = V_1'$, $V_2 = V_2'$, $V_3 = V_3' \cdots$, the theorem follows from § 419.

438. Convex Closed Curves. We assume that in *a convex closed curve* (§ 202) it is possible to inscribe a sequence of polygons $P_1, P_2, P_3, \cdots$, having perimeters $p_1, p_2, p_3, \cdots$ and areas $A_1, A_2, A_3, \cdots$ with limits p and A respectively, and to circumscribe a sequence of polygons $P_1', P_2', P_3', \cdots$, having perimeters $p_1', p_2', p_3', \cdots$ and areas $A_1', A_2', A_3', \cdots$ with limits p' and A' respectively, such that $p = p'$ and $A = A'$.

These limits p and A we now define as **the perimeter and the area respectively of the curve.**

APPLICATION OF LIMITS TO THE CYLINDER

439. The Cylinder. Given any cylinder with a convex right cross-section and an element e. In this cross-section inscribe a sequence P_1, P_2, P_3, ⋯ of polygons, as in § 438, with perimeters p_1, p_2, p_3, ⋯ and areas A_1, A_2, A_3, ⋯, thus defining the perimeter p and the area A of the cross-section.

Consider a set of prisms inscribed in this cylinder, of which P_1, P_2, P_3, ⋯ are right cross-sections.

Then the areas and the volumes of these prisms are respectively p_1e, p_2e, p_3e, ⋯ and A_1e, A_2e, A_3e, ⋯.

The lateral area and the volume of the cylinder are now defined as the limits of these sequences. But by § 428 these limits are equal respectively to pe and Ae.

Hence, we have the theorem:

440. THEOREM X. *The lateral area of a cylinder is the product of an element, and the perimeter of a right section and its volume is the product of an element and the area of a right section.*

EXERCISES

1. Prove as above that the volume of any convex cylinder is equal to the product of its altitude and the area of its base.

2. Prove that the lateral area of a right circular cone is equal to half the product of the slant height and the perimeter of its base.

3. Prove that the volume of any convex cone is equal to one third the product of its altitude and the area of its base.

Suggestion. The treatment required in these exercises is a very close paraphrase of the definitions and proof given in § 439. Observe that we cannot begin to make a proof until we have *defined* the subject matter of the theorem. That is, we must first define **the areas and volumes in question.**

APPLICATION OF LIMITS TO THE SPHERE

441. The Sphere. Through two points P and Q of a sphere pass a great circle forming a hemisphere with center O.
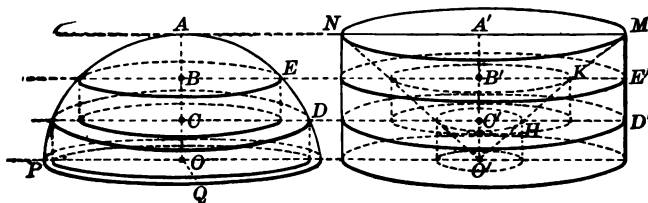
Divide OA, the radius perpendicular to the plane of the great circle, into the equal parts OC, CB, and BA. Through C and B pass planes parallel to the plane of POQ, meeting the sphere in points D and E, respectively.

. Construct right circular cylinders with axes OC and CB and radii CD and BE. Denote by V_1 the sum of the volumes of these cylinders.

Now divide the radius OA into six equal parts and construct five cylinders in the same manner as above. Let the sum of these volumes be V_2.

Continuing in this manner, each time dividing OA into twice as many equal parts as in the preceding, we obtain a sequence of sets of cylinders and a corresponding sequence $V_1, V_2, V_3, \cdots$ of volumes.

We now define **the volume of the hemisphere as the limit of the sequence $V_1, V_2, V_3, \cdots$**.

442. Construct a right circular cylinder with its base in the plane of POQ and with radius and altitude both equal to OA.

Denote by F the figure formed by the lower base of the cylinder, its lateral surface and the lateral surface of the cone whose base is the upper base of the cylinder and whose vertex is at O'. (See the right-hand figure.)

Draw segments $O'M$ and $O'N$. Let the planes through C and B cut $O'A'$ in C' and B' and $O'M$ in H and K.

Now form the cylinder $O'C'H$ whose axis is $O'C'$ and whose radius is $C'H$. Likewise form the cylinder $C'B'K$.

Let V_1' denote the sum of the volumes of $O'C'D'$ and $C'B'E'$ minus the sum of the volumes of $O'C'H$ and $C'B'K$.

In a similar manner, using the planes which divide OA and hence $O'A'$ into six equal parts, we form another set of five cylinders, the sum of whose volumes minus that of the smaller inside cylinders we denote by V_2'.

Continuing in this manner, we obtain a sequence of volumes V_1', V_2', V_3', $\cdots$ whose limit V' we define as **the volume of the given figure** F.

We now prove that $V_1 = V_1'$, $V_2 = V_2'$, $\cdots$.

Denote OA by r, and note that $O'B' = B'K$.

(1) Vol. $C'B'E'$ — Vol. $C'B'K = \pi C'B' (\overline{B'E'}^2 - \overline{B'K}^2) = \pi C'B'(r^2 - \overline{O'B'}^2)$.

(2) Vol. $CBE = \pi CB \cdot \overline{BE}^2 = \pi CB_(r^2 - \overline{OB}^2)$, since $\overline{BE}^2 = r^2 - \overline{OB}^2$. But $OB = O'B'$ and $CB = C'B'$.

Hence, Vol. $CBE = \pi C'B'(r^2 - \overline{O'B'}^2)$.

Hence, Vol. $CBE =$ Vol. $C'B'E'$ — Vol. $C'B'K$.

Similarly we show that

$$\text{Vol. } OCD = \text{Vol. } O'C'D' - \text{Vol. } O'C'H.$$

Hence, $V_1 = V_1'$. In like manner $V_2 = V_2'$, $V_3 = V_3'$, $\cdots$.

Hence, $V = V'$, since they are the limits of the same sequences.

But the volume of the cylinder $O'A'M$ is πr^3 and of the cone whose volume was subtracted, $\frac{1}{3}\pi r^3$. That is, the volume of F is $\frac{2}{3}\pi r^3$, and hence that of the hemisphere is $\frac{2}{3}\pi r^2$.

Hence, we have the theorem:

443. THEOREM XI.   *The volume of the sphere is $\frac{4}{3}\pi r^3$.*

Note that the above proof consists essentially in showing that the area of the circle BE is equal to that of the ring between the circles $B'E'$ and $B'K$, and that the area of the circle CD is equal to that of the ring between $C'D'$ and $C'H$ and so on.

Indeed, this theorem and also that of § 437 are special cases of what is known as **Cavalieri's Theorem.**

444. THEOREM XII. *If two solid figures are regarded as resting on the same plane b, and if in every plane parallel to b the sections of the two figures have equal areas, the figures have equal volumes.*

The proof of this general theorem is more difficult than any thus far given, inasmuch as it involves sequences which *oscillate;* that is, which are neither constantly increasing nor constantly decreasing.

445. The Area of the Sphere. About a sphere of radius r construct a sequence of circumscribed polyhedrons such that the largest face in each polyhedron becomes as small as we please when we proceed along the sequence. Let $s_1, s_2, s_3, \cdots$ be the total surfaces of these polyhedrons. This forms a decreasing sequence with limit S which we define as **the surface of the sphere.**

The volumes of these polyhedrons will be $\frac{1}{3}rs_1, \frac{1}{3}rs_2, \frac{1}{3}rs_3, \cdots$.

Then the volume V of the sphere is defined[1] **as the limit of this sequence of volumes.**

Hence, by § 428, $V = \frac{1}{3}rS$. But by § 443, $V = \frac{4}{3}\pi r^3$.

Then $S = \frac{3}{r} \cdot \frac{4}{3}\pi r^3 = 4\pi r^2$.

Hence, we have the theorem:

446. THEOREM XIII. *The area of the sphere is $4\pi r^2$.*

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[1] This definition can be shown to be consistent with that of § 441.

EXERCISES ON LIMITS

1. In addition to those which are found in the text give other examples of infinite sequences which do not determine definite numbers.

2. Give two increasing sequences which determine the number 3. Show that the theorem of § 420 applies and proves that these sequences determine the same number.

3. Give two decreasing sequences each of which determines the number 5. Apply § 421 to show that these sequences determine the same number.

4. State fully the relation between a bounded increasing sequence and the number determined by it. State also the relations between a bounded decreasing sequence and the number determined by it.

5. State fully what is meant by "a limit of a sequence" both for increasing and decreasing sequences.

6. Given two incommensurable segments AB and CD. Lay off on the line AB a *decreasing* sequence of segments, each of which is commensurable with CD, such that the limit of the sequence is the segment AB.

7. If a_1, a_2, a_3, $\cdots$ is an increasing sequence defining the number 4, prove that $3\,a_1$, $3\,a_2$, $3\,a_3$, $\cdots$ defines the number $3 \times 4 = 12$.

8. If a_1, a_2, a_3, and b_1, b_2, b_3, $\cdots$ are increasing sequences defining the numbers 3 and 5, show that the sequence a_1b_1, a_2b_2, a_3b_3, $\cdots$ defines the number 15.

9. Describe in more detail the meaning of the ratio of two incommensurable arcs as indicated in § 425. Show that this ratio is independent of the sequence of units of measurement used, so long as the limit of this sequence is zero.

10. Treat the ratio of two incommensurable angles in a manner similar to the treatment of arcs in the preceding exercise.

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